

ASYMPTOTIC BEHAVIOR OF SOLUTIONS TOWARD A MULTIWAVE PATTERN TO THE CAUCHY PROBLEM FOR THE SCALAR CONSERVATION LAW WITH DEGENERATE FLUX AND VISCOSITY

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Abstract. In this paper, we investigate the asymptotic behavior of solutions toward a multiwave pattern of the Cauchy problem for the scalar viscous conservation law where the far field states are prescribed. Especially, we deal with the case when the flux function is convex or concave but linearly degenerate on some interval, and also the viscosity is a nonlinearly degenerate one (p -Laplacian type viscosity). When the corresponding Riemann problem admits a Riemann solution which consists of rarefaction waves and contact discontinuity, it is proved that the solution of the Cauchy problem tends toward the linear combination of the rarefaction waves and contact wave for p -Laplacian type viscosity as the time goes to infinity. This is the first result concerning the asymptotics toward multiwave pattern for the Cauchy problem of the scalar conservation law with nonlinear viscosity. The proof is given by a technical energy methods and the careful estimates for the interactions between the nonlinear waves.

Key words. viscous conservation law, asymptotic behavior, nonlinearly degenerate viscosity, linearly degenerate flux, multiwave pattern, rarefaction wave, viscous contact wave

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1. Introduction and main theorem. In this paper, we shall consider the asymptotic behavior of solutions for one-dimensional scalar conservation law with a nonlinearly degenerate viscosity (p -Laplacian type viscosity with $p > 1$)

$$\begin{cases} \partial_t u + \partial_x(f(u)) = \mu \partial_x \left(|\partial_x u|^{p-1} \partial_x u \right) & (t > 0, x \in \mathbb{R}), \\ u(0, x) = u_0(x) & (x \in \mathbb{R}), \\ \lim_{x \rightarrow \pm\infty} u(t, x) = u_{\pm} & (t \geq 0). \end{cases} \quad (1.1)$$

Here, $u = u(t, x)$ denotes the unknown function of $t > 0$ and $x \in \mathbb{R}$, the so-called conserved quantity, $f = f(u)$ is the flux function depending only on u , μ is the viscosity coefficient, u_0 is the given initial data, and constants $u_{\pm} \in \mathbb{R}$ are the prescribed far field states. We suppose the given flux $f = f(u)$ is a C^1 -function satisfying $f(0) = f'(0) = 0$, μ is a positive constant and far field states u_{\pm} satisfy $u_- < u_+$ without loss of generality.

We are interested in the asymptotic behavior and its precise estimates in time of the global solution to the Cauchy problem (1.1). Especially, one of the keys of the study is to investigate the influence of the shape of the flux function $f(u)$ and the far field states u_{\pm} on the asymptotic behavior. It can be expected that the large-time behavior is closely related to the weak solution (“Riemann solution”) of the corresponding Riemann problem (cf. [13], [27]) for the non-viscous hyperbolic part of (1.1):

$$\begin{cases} \partial_t u + \partial_x(f(u)) = 0 & (t > 0, x \in \mathbb{R}), \\ u(0, x) = u_0^R(x) & (x \in \mathbb{R}), \end{cases} \quad (1.2)$$

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where u_0^R is the Riemann data defined by

$$u_0^R(x) = u_0^R(x; u_-, u_+) := \begin{cases} u_- & (x < 0), \\ u_+ & (x > 0). \end{cases}$$

In fact, for the usual linear viscosity case:

$$\begin{cases} \partial_t u + \partial_x(f(u)) = \mu \partial_x^2 u & (t > 0, x \in \mathbb{R}), \\ u(0, x) = u_0(x) & (x \in \mathbb{R}), \\ \lim_{x \rightarrow \pm\infty} u(t, x) = u_{\pm} & (t \geq 0), \end{cases} \quad (1.3)$$

when the smooth flux function f is genuinely nonlinear on the whole space \mathbb{R} , i.e., $f''(u) \neq 0$ ($u \in \mathbb{R}$), Il'in-Oleřnik [10] showed the following: if $f''(u) > 0$ ($u \in \mathbb{R}$), that is, the Riemann solution consists of a single rarefaction wave solution, the global solution in time of the Cauchy problem (1.3) tends toward the rarefaction wave; if $f''(u) < 0$ ($u \in \mathbb{R}$), that is, the Riemann solution consists of a single shock wave solution, the global solution of the Cauchy problem (1.3) does the corresponding smooth traveling wave solution (“viscous shock wave”) of (1.3) with a spacial shift (cf. [9]). More generally, in the case of the flux functions which are not uniformly genuinely nonlinear, when the Riemann solution consists of a single shock wave satisfying Oleřnik’s shock condition, Matsumura-Nishihara [19] showed the asymptotic stability of the corresponding viscous shock wave. However, when we consider the circumstances where the Riemann solution generically forms a pattern of multiple nonlinear waves which consists of rarefaction waves, shock waves and waves of contact discontinuity (refer to [14]), there had been no results about the asymptotics toward the multiwave pattern. Recently, Matsumura-Yoshida [20] proved the asymptotics toward a multiwave pattern of the superposition of the rarefaction waves and a self-similar solution (“viscous contact wave”) which is corresponded to the wave of the contact discontinuity. Namely, they investigated the case where the flux function f is smooth and genuinely nonlinear (that is, f is convex function or concave function) on the whole \mathbb{R} except a finite interval $I := (a, b) \subset \mathbb{R}$, and linearly degenerate on I , that is,

$$\begin{cases} f''(u) > 0 & (u \in (-\infty, a] \cup [b, +\infty)), \\ f''(u) = 0 & (u \in (a, b)). \end{cases} \quad (1.4)$$

For the flux function satisfying (1.4), the corresponding Riemann solution does form multiwave pattern which consists of the contact discontinuity with the jump from $u = a$ to $u = b$ and the rarefaction waves, depending on the choice of a , b , u_- and u_+ . Thanks to that the cases in which the interval (a, b) is disjoint from the interval (u_-, u_+) are similar as in the case the flux function f is genuinely nonlinear on the whole space \mathbb{R} , and the case $u_- < a < u_+ < b$ is the same as that for $a < u_- < b < u_+$, we may only consider the typical cases

$$a < u_- < b < u_+ \quad \text{or} \quad u_- < a < b < u_+. \quad (1.5)$$

Under the conditions (1.4) and (1.5), they have shown the unique global solution in time to the Cauchy problem (1.3) tends uniformly in space toward the multiwave pattern of the combination of the viscous contact wave and the rarefaction waves as the time goes to infinity. It should be noted that the rarefaction wave which connects

the far field states u_- and u_+ ($u_{\pm} \in (-\infty, a]$ or $u_{\pm} \in [b, \infty)$) is explicitly given by

$$u = u^r \left(\frac{x}{t}; u_-, u_+ \right) := \begin{cases} u_- & (x \leq \lambda(u_-)t), \\ (\lambda)^{-1} \left(\frac{x}{t} \right) & (\lambda(u_-)t \leq x \leq \lambda(u_+)t), \\ u_+ & (x \geq \lambda(u_+)t), \end{cases} \quad (1.6)$$

where $\lambda(u) := f'(u)$, and the viscous contact wave which connects u_- and u_+ ($u_{\pm} \in [a, b]$) is given by an exact solution of the linear convective heat equation

$$\partial_t u + \tilde{\lambda} \partial_x u = \mu \partial_x^2 u \quad \left(\tilde{\lambda} := \frac{f(b) - f(a)}{b - a}, t > 0, x \in \mathbb{R} \right) \quad (1.7)$$

which has the form

$$u = U \left(\frac{x - \tilde{\lambda} t}{\sqrt{t}}; u_-, u_+ \right)$$

where $U \left(\frac{x}{\sqrt{t}}; u_-, u_+ \right)$ is explicitly defined by

$$U \left(\frac{x}{\sqrt{t}}; u_-, u_+ \right) := u_- + \frac{u_+ - u_-}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{\sqrt{4\mu t}}} e^{-\xi^2} d\xi \quad (t > 0, x \in \mathbb{R}). \quad (1.8)$$

Yoshida [28] also obtained the precise decay properties for the asymptotics (cf. [5]). In the proof of them, the a priori energy estimates acquired by an L^2 -energy method and careful estimates for the terms of nonlinear interactions of the viscous contact wave and the rarefaction waves.

The aim of the present paper is to extend the results in the previous study in [20] to the case where the viscosity is of p -Laplacian type (the related problems are studied in [4], [21], [22] and so on). For this case, a main difficulty arises from the fact that when $u_{\pm} \in [a, b]$, the asymptotic state is expected to be a self-similar type solution of a nonlinearly degenerate convective heat equation which may need the more subtle treatment than the Gaussian kernel type one (1.8) of the equation (1.7). There is only one result for the asymptotic behavior for the problem (1.1) in the case where the flux function is genuinely nonlinear on the whole space \mathbb{R} . Namely, Matsumura-Nishihara [18] proved the asymptotics which tends toward a single rarefaction wave by using the L^2 and L^p -energy estimates. We then consider the case where the flux function is given as (1.4) and the far field states as (1.5). We expect the asymptotic behavior of solutions to the Cauchy problem (1.1) to be similar as in [20]. In more detail, under the conditions (1.4) and (1.5), if the far field states u_{\pm} satisfy $u_{\pm} \in (-\infty, a]$ or $u_{\pm} \in [b, \infty)$, the asymptotic state of the solutions to the Cauchy problem (1.1) should be the rarefaction wave (1.6) which connects u_- and u_+ , and if the far field states u_{\pm} satisfy $u_{\pm} \in [a, b]$, the one should be the “contact wave for p -Laplacian type viscosity” which connects u_- and u_+ , which is given by an exact solution of the following p -Laplacian evolution equation

$$\partial_t u + \tilde{\lambda} \partial_x u = \mu \partial_x \left(|\partial_x u|^{p-1} \partial_x u \right) \quad \left(\tilde{\lambda} := \frac{f(b) - f(a)}{b - a}, t > 0, x \in \mathbb{R} \right). \quad (1.9)$$

In order to look for an exact solution, especially self-similar type solution, we differentiate the evolution equation (1.9) with respect to x and we have the following porous

medium equation with the convection term

$$\partial_t v + \tilde{\lambda} \partial_x v = \mu \partial_x^2 \left(|v|^{p-1} v \right), \quad (1.10)$$

where $v := \partial_x u$. Barenblatt [1], Zel'dovič-Kompaneec [29] and Pattle [25] (see also [2], [8], [11]) introduced the following Cauchy problem of the porous medium equation

$$\begin{cases} \partial_t v = \mu \partial_x^2 \left(|v|^{p-1} v \right) & (t > -1, x \in \mathbb{R}), \\ v(-1, x) = (u_+ - u_-) \delta(x) & (x \in \mathbb{R}; u_- < u_+), \\ \lim_{x \rightarrow \pm\infty} v(t, x) = 0 & (t \geq -1), \end{cases} \quad (1.11)$$

where $\delta(x)$ is the Dirac δ -distribution. They obtained the Barenblatt-Kompaneec-Zel'dovič solution

$$v(t, x) := \frac{1}{(1+t)^{\frac{1}{p+1}}} \left(\left(A - B \left(\frac{x}{(1+t)^{\frac{1}{p+1}}} \right)^2 \right) \vee 0 \right)^{\frac{1}{p-1}}, \quad (1.12)$$

$$\begin{cases} A = A_{p, \mu, u_{\pm}} := \left(\frac{(p-1)(u_+ - u_-)}{8\mu p(p+1) \left(\int_0^{\frac{\pi}{2}} (\sin \theta)^{\frac{p+1}{p-1}} d\theta \right)^2} \right)^{\frac{p-1}{p+1}}, \\ B = B_{p, \mu} := \frac{p-1}{2\mu p(p+1)}, \\ 2A^{\frac{p+1}{2(p-1)}} B^{-\frac{1}{2}} \int_0^{\frac{\pi}{2}} (\sin \theta)^{\frac{p+1}{p-1}} d\theta = u_+ - u_-, \end{cases}$$

where the symbol “ \vee ” is defined as $a \vee b := \max\{a, b\}$. Thus, when we define by using the solution (1.12) as

$$\begin{aligned} U \left(\frac{x}{(1+t)^{\frac{1}{p+1}}}; u_-, u_+ \right) &:= u_- + \int_{-\infty}^x v(t, y) dy \\ &= u_- + \int_{-\infty}^{\frac{x}{(1+t)^{\frac{1}{p+1}}}} \left((A - B\xi^2) \vee 0 \right)^{\frac{1}{p-1}} d\xi \\ &\quad (t > -1, x \in \mathbb{R}), \end{aligned} \quad (1.13)$$

and change the variable as $1+t \mapsto t > 0$ and $x \mapsto x - \tilde{\lambda}t$ in this order, we have a desired candidate of the asymptotic state as

$$U \left(\frac{x - \tilde{\lambda}t}{t^{\frac{1}{p+1}}}; u_-, u_+ \right) = u_- + \int_{-\infty}^{\frac{x - \tilde{\lambda}t}{t^{\frac{1}{p+1}}}} \left((A - B\xi^2) \vee 0 \right)^{\frac{1}{p-1}} d\xi \quad (1.14)$$

which is said to be “contact wave for p -Laplacian type viscosity”. Now we are ready to state our main result.

Theorem 1.1 (Main Theorem). *Let the flux function f satisfy (1.4) and the far field states u_{\pm} (1.5). Assume that the initial data satisfies $u_0 - u_0^R \in L^2$ and $\partial_x u_0 \in L^{p+1}$. Then the Cauchy problem (1.1) with $p > 1$ has a unique global solution in time $u = u(t, x)$ satisfying*

$$\begin{cases} u - u_0^R \in C^0([0, \infty); L^2) \cap L^\infty(\mathbb{R}^+; L^2), \\ \partial_x u \in L^\infty(\mathbb{R}^+; L^{p+1}), \\ \partial_t u \in L^\infty(\mathbb{R}^+; L^{p+1}), \\ \partial_x (|\partial_x u|^{p-1} \partial_x u) \in L^2(\mathbb{R}_t^+ \times \mathbb{R}_x), \end{cases}$$

and the asymptotic behavior

$$\lim_{t \rightarrow \infty} \sup_{x \in \mathbb{R}} |u(t, x) - U_{multi}(t, x; u_-, u_+)| = 0,$$

where $U_{multi}(t, x) = U_{multi}(t, x; u_-, u_+)$ is defined as follows: in the case $a < u_- < b < u_+$,

$$U_{multi}(t, x) := U\left(\frac{x - \tilde{\lambda} t}{t^{\frac{1}{p+1}}}; u_-, b\right) + u^r\left(\frac{x}{t}; b, u_+\right) - b$$

and, in the case $u_- < a < b < u_+$,

$$U_{multi}(t, x) := u^r\left(\frac{x}{t}; u_-, a\right) - a + U\left(\frac{x - \tilde{\lambda} t}{t^{\frac{1}{p+1}}}; a, b\right) + u^r\left(\frac{x}{t}; b, u_+\right) - b.$$

The main theorem is proved by using a technical energy method with the aid of the maximum principle, and the careful estimates of the nonlinear interactions between the nonlinear waves, that is, the rarefaction waves and the contact wave for the p -Laplacian type viscosity.

This paper is organized as follows. In Section 2, we shall prepare the basic properties of the rarefaction wave and the contact wave for p -Laplacian type viscosity. In Section 3, we reduce the problem to an essential case (similarly in [20], [28]) and reformulate the problem in terms of the deviation from the asymptotic state, that is, the superposition of the nonlinear waves. Following the arguments in [18], we show the global existence of the solution to the reformulated problem and the energy estimates which are depending on the time. In order to show the asymptotics, in Section 4 and Section 5, we establish the uniform energy estimates in time by using a very technical energy method and careful estimates of the interactions between the nonlinear waves. Finally, in Section 6, we prove the asymptotic behavior by utilizing the uniform energy estimates in Section 4 and Section 5.

Some Notation. We denote by C generic positive constants unless they need to be distinguished. In particular, use $C(\alpha, \beta, \dots)$ or $C_{\alpha, \beta, \dots}$ when we emphasize the dependency on α, β, \dots , and \mathbb{R}^+ as $\mathbb{R}^+ := (0, \infty)$. We also use the Friedrichs mollifier ρ_δ^* , where, $\rho_\delta(x) := \frac{1}{\delta} \rho\left(\frac{x}{\delta}\right)$ with

$$\begin{aligned} \rho &\in C_0^\infty(\mathbb{R}), \quad \rho(x) \geq 0 \quad (x \in \mathbb{R}), \\ \text{supp}\{\rho\} &\subset \{x \in \mathbb{R} \mid |x| \leq 1\}, \quad \int_{-\infty}^{\infty} \rho(x) dx = 1, \end{aligned}$$

and $\rho_\delta * f$ denote the convolution. For function spaces, $L^p = L^p(\mathbb{R})$ and $H^k = H^k(\mathbb{R})$ denote the usual Lebesgue space and k -th order Sobolev space on the whole space \mathbb{R} with norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_{H^k}$, respectively. We also define the bounded C^m -class \mathcal{B}^m as follows

$$f \in \mathcal{B}^m(\Omega) \iff f \in C^m(\Omega), \sup_{\Omega} \sum_{k=0}^m |D^k f| < \infty$$

for $m < \infty$ and

$$f \in \mathcal{B}^\infty(\Omega) \iff \forall n \in \mathbb{N}, f \in C^n(\Omega), \sup_{\Omega} \sum_{k=0}^n |D^k f| < \infty$$

where $\Omega \subset \mathbb{R}^d$ and D^k denote the all of k -th order derivatives.

2. Preliminaries. In this section, we shall arrange the several lemmas concerning with the basic properties of the rarefaction wave and the viscous contact wave for accomplishing the proof of the main theorem. Since the rarefaction wave u^r is not smooth enough, we need some smooth approximated one as in the previous works in [6], [16], [17], [20]. We start with the well-known arguments on u^r and the method of constructing its smooth approximation. We first consider the rarefaction wave solution w^r to the Riemann problem for the non-viscous Burgers equation

$$\begin{cases} \partial_t w + \partial_x \left(\frac{1}{2} w^2 \right) = 0 & (t > 0, x \in \mathbb{R}), \\ w(0, x) = w_0^R(x; w_-, w_+) := \begin{cases} w_+ & (x > 0), \\ w_- & (x < 0), \end{cases} \end{cases} \quad (2.1)$$

where $w_\pm \in \mathbb{R}$ ($w_- < w_+$) are the prescribed far field states. The unique global weak solution $w = w^r\left(\frac{x}{t}; w_-, w_+\right)$ of (2.1) is explicitly given by

$$w^r\left(\frac{x}{t}; w_-, w_+\right) := \begin{cases} w_- & (x \leq w_- t), \\ \frac{x}{t} & (w_- t \leq x \leq w_+ t), \\ w_+ & (x \geq w_+ t). \end{cases} \quad (2.2)$$

Next, under the condition $f''(u) > 0$ ($u \in \mathbb{R}$) and $u_- < u_+$, the rarefaction wave solution $u = u^r\left(\frac{x}{t}; u_-, u_+\right)$ of the Riemann problem (1.2) for hyperbolic conservation law is exactly given by

$$u^r\left(\frac{x}{t}; u_-, u_+\right) = (\lambda)^{-1} \left(w^r\left(\frac{x}{t}; \lambda_-, \lambda_+\right) \right) \quad (2.3)$$

which is nothing but (1.6), where $\lambda_\pm := \lambda(u_\pm) = f'(u_\pm)$. We define a smooth approximation of $w^r\left(\frac{x}{t}; w_-, w_+\right)$ by the unique classical solution

$$w = w(t, x; w_-, w_+) \in \mathcal{B}^\infty([0, \infty) \times \mathbb{R})$$

to the Cauchy problem for the following non-viscous Burgers equation

$$\begin{cases} \partial_t w + \partial_x \left(\frac{1}{2} w^2 \right) = 0 & (t > 0, x \in \mathbb{R}), \\ w(0, x) = w_0(x) := \frac{w_- + w_+}{2} + \frac{w_+ - w_-}{2} \tanh x & (x \in \mathbb{R}), \end{cases} \quad (2.4)$$

By using the method of characteristics, we get the following formula

$$\begin{cases} w(t, x) = w_0(x_0(t, x)) = \frac{\lambda_- + \lambda_+}{2} + \frac{\lambda_+ - \lambda_-}{2} \tanh(x_0(t, x)), \\ x = x_0(t, x) + w_0(x_0(t, x)) t. \end{cases} \quad (2.5)$$

We also note the assumption of the flux function f to be $\lambda'(u) \left(= \frac{d^2 f}{du^2}(u) \right) > 0$.

Now we summarize the results for the smooth approximation $w(t, x; w_-, w_+)$ in the next lemma. Since the proof is given by the direct calculation as in [17], we omit it.

Lemma 2.1. *Assume that the far field states satisfy $w_- < w_+$. Then the classical solution $w(t, x) = w(t, x; w_-, w_+)$ given by (2.4) satisfies the following properties:*

- (1) $w_- < w(t, x) < w_+$ and $\partial_x w(t, x) > 0$ ($t > 0, x \in \mathbb{R}$).
- (2) For any $1 \leq q \leq \infty$, there exists a positive constant C_q such that

$$\begin{aligned} \|\partial_x w(t)\|_{L^q} &\leq C_q (1+t)^{-1+\frac{1}{q}} \quad (t \geq 0), \\ \|\partial_x^2 w(t)\|_{L^q} &\leq C_q (1+t)^{-1} \quad (t \geq 0). \end{aligned}$$

$$(3) \lim_{t \rightarrow \infty} \sup_{x \in \mathbb{R}} \left| w(t, x) - w^r\left(\frac{x}{t}\right) \right| = 0.$$

We define the approximation for the rarefaction wave $u^r\left(\frac{x}{t}; u_-, u_+\right)$ by

$$U^r(t, x; u_-, u_+) := (\lambda)^{-1}(w(t, x; \lambda_-, \lambda_+)). \quad (2.6)$$

Then we have the next lemma as in the previous works (cf. [6], [16], [17], [20]).

Lemma 2.2. *Assume that the far field states satisfy $u_- < u_+$, and the flux function $f \in C^3(\mathbb{R})$, $f''(u) > 0$ ($u \in [u_-, u_+]$). Then we have the following properties:*

- (1) $U^r(t, x)$ defined by (2.6) is the unique C^2 -global solution in space-time of the Cauchy problem

$$\begin{cases} \partial_t U^r + \partial_x (f(U^r)) = 0 & (t > 0, x \in \mathbb{R}), \\ U^r(0, x) = (\lambda)^{-1} \left(\frac{\lambda_- + \lambda_+}{2} + \frac{\lambda_+ - \lambda_-}{2} \tanh x \right) & (x \in \mathbb{R}), \\ \lim_{x \rightarrow \pm \infty} U^r(t, x) = u_{\pm} & (t \geq 0). \end{cases}$$

- (2) $u_- < U^r(t, x) < u_+$ and $\partial_x U^r(t, x) > 0$ ($t > 0, x \in \mathbb{R}$).
- (3) For any $1 \leq q \leq \infty$, there exists a positive constant C_q such that

$$\begin{aligned} \|\partial_x U^r(t)\|_{L^q} &\leq C_q (1+t)^{-1+\frac{1}{q}} \quad (t \geq 0), \\ \|\partial_x^2 U^r(t)\|_{L^q} &\leq C_q (1+t)^{-1} \quad (t \geq 0). \end{aligned}$$

$$(4) \lim_{t \rightarrow \infty} \sup_{x \in \mathbb{R}} \left| U^r(t, x) - u^r\left(\frac{x}{t}\right) \right| = 0.$$

- (5) For any $\epsilon \in (0, 1)$, there exists a positive constant C_ϵ such that

$$|U^r(t, x) - u_+| \leq C_\epsilon (1+t)^{-1+\epsilon} e^{-\epsilon|x-\lambda_+ t|} \quad (t \geq 0, x \geq \lambda_+ t).$$

(6) For any $\epsilon \in (0, 1)$, there exists a positive constant C_ϵ such that

$$|U^r(t, x) - u_-| \leq C_\epsilon(1+t)^{-1+\epsilon} e^{-\epsilon|x-\lambda_-t|} \quad (t \geq 0, x \leq \lambda_-t).$$

(7) For any $\epsilon \in (0, 1)$, there exists a positive constant C_ϵ such that

$$\left| U^r(t, x) - u^r\left(\frac{x}{t}\right) \right| \leq C_\epsilon(1+t)^{-1+\epsilon} \quad (t \geq 1, \lambda_-t \leq x \leq \lambda_+t).$$

(8) For any $(\epsilon, q) \in (0, 1) \times [1, \infty]$, there exists a positive constant $C_{\epsilon, q}$ such that

$$\left\| U^r(t, \cdot) - u^r\left(\frac{\cdot}{t}\right) \right\|_{L^q} \leq C_{\epsilon, q}(1+t)^{-1+\frac{1}{q}+\epsilon} \quad (t \geq 0).$$

Because the proofs of (1) to (4) are given in [17], (5) to (7) are in [20] and (8) is in [28], we omit the proofs here.

We also prepare the next lemma for the properties of the contact wave for p -Laplacian type viscosity $U\left(\frac{x}{t^{\frac{1}{p+1}}}; u_-, u_+\right)$ defined by (1.11). In the following, we abbreviate “contact wave for p -Laplacian type viscosity” to “viscous contact wave”. Substituting (1.12) into (1.13), we rewrite the viscous contact wave as

$$\begin{aligned} U(t, x) &= U\left(\frac{x}{t^{\frac{1}{p+1}}}; u_-, u_+\right) \\ &= u_+ - \int_x^\infty \frac{1}{t^{\frac{1}{p+1}}} \left(\left(A - B \left(\frac{y}{t^{\frac{1}{p+1}}} \right)^2 \right) \vee 0 \right)^{\frac{1}{p-1}} dy, \end{aligned} \quad (2.7)$$

where

$$\begin{cases} A = A_{p, \mu, u_\pm} := \left(\frac{(p-1)(u_+ - u_-)}{8\mu p(p+1) \left(\int_0^{\frac{\pi}{2}} (\sin \theta)^{\frac{p+1}{p-1}} d\theta \right)^2} \right)^{\frac{p-1}{p+1}}, \\ B = B_{p, \mu} := \frac{p-1}{2\mu p(p+1)}, \\ 2A^{\frac{p+1}{2(p-1)}} B^{-\frac{1}{2}} \int_0^{\frac{\pi}{2}} (\sin \theta)^{\frac{p+1}{p-1}} d\theta = u_+ - u_-. \end{cases}$$

Then, we have the next lemma. Because the proofs are very elementary, we omit the proofs.

Lemma 2.3. For any $p > 1$ and $u_\pm \in \mathbb{R}$, we have the following:

(i) U defined by (1.11) satisfies

$$U \in \mathcal{B}^1((0, \infty) \times \mathbb{R}) \setminus C^2 \left(\left\{ (t, x) \in \mathbb{R}^+ \times \mathbb{R} \mid x = \pm \sqrt{\frac{A}{B}} t^{\frac{1}{p+1}} \right\} \right),$$

and is a self-similar type strong solution of the Cauchy problem

$$\begin{cases} \partial_t U - \mu \partial_x (|\partial_x U|^{p-1} \partial_x U) = 0 & (t > 0, x \in \mathbb{R}), \\ U(0, x) = u_0^R(x; u_-, u_+) = \begin{cases} u_- & (x < 0), \\ u_+ & (x > 0), \end{cases} \\ \lim_{x \rightarrow \pm\infty} U(t, x) = u_{\pm} & (t \geq 0). \end{cases}$$

(ii) For $t > 0$ and $x \in \mathbb{R}$,

$$\begin{cases} U(t, x) = u_-, & \left(x \leq -\sqrt{\frac{A}{B}} t^{\frac{1}{p+1}}\right), \\ u_- < U(t, x) < u_+, \quad \partial_x U(t, x) > 0 & \left(-\sqrt{\frac{A}{B}} t^{\frac{1}{p+1}} < x < \sqrt{\frac{A}{B}} t^{\frac{1}{p+1}}\right), \\ U(t, x) = u_+, & \left(x \geq \sqrt{\frac{A}{B}} t^{\frac{1}{p+1}}\right). \end{cases}$$

(iii) It holds that for any $1 \leq q < \infty$,

$$\|\partial_x U(t)\|_{L^q} = C_1(A, B; p, q) t^{-\frac{q-1}{(p+1)q}} \quad (t > 0)$$

where

$$C_1(A, B; p, q) := \left(2A^{\frac{p+2q-1}{2(p-1)}} B^{-\frac{1}{2}} \int_0^{\frac{\pi}{2}} (\sin \theta)^{\frac{q}{p-1}} d\theta\right)^{\frac{1}{q}}.$$

If $q = \infty$, we have

$$\|\partial_x U(t)\|_{L^\infty} = (2A)^{\frac{1}{p-1}} t^{-\frac{1}{p+1}} \quad (t > 0).$$

(iv) It holds that for any $1 \leq q < \frac{p-1}{p-2}$ with $p > 2$, or any $1 \leq q < \infty$ with $1 < p \leq 2$,

$$\|\partial_x^2 U(t)\|_{L^q} = C_2(A, B; p, q) t^{-\frac{2q-1}{(p+1)q}} \quad (t > 0)$$

where

$$C_2(A, B; p, q) := \left(2 \left(\frac{2A^{-\frac{p-2}{p-1}} B}{p-1}\right)^q \left(\frac{B}{A}\right)^{-\frac{q+1}{2}} \int_0^{\frac{\pi}{2}} (\sin \theta)^{-\frac{2(p-2)q}{p-1}+1} (\cos \theta)^q d\theta\right)^{\frac{1}{q}}.$$

If $1 < p \leq 2$, for $q = \infty$, we have

$$\|\partial_x^2 U(t)\|_{L^\infty} = \frac{2A^{\frac{p-2}{p-1}} B}{p-1} \left(\frac{B}{A}\right)^{-\frac{1}{2}} t^{-\frac{2}{p+1}} \quad (t > 0).$$

(v) It holds that

$$\left\| \partial_x \left(|\partial_x U|^{p-1} \partial_x U \right) (t) \right\|_{L^2} = C_3(A, B; p) t^{-\frac{2p+1}{2(p+1)}} \quad (t > 0)$$

where

$$C_3(A, B; p) := \left(2 \left(\frac{2B^p}{p-1}\right)^2 \left(\frac{B}{A}\right)^{-\frac{3p-7}{2(p-1)}} \int_0^{\frac{\pi}{2}} (\sin \theta)^{\frac{p+3}{p-1}} (\cos \theta)^2 d\theta\right)^{\frac{1}{2}}.$$

(vi) $\lim_{t \rightarrow \infty} \sup_{x \in \mathbb{R}} |U(1+t, x) - U(t, x)| = 0$.

3. Reformulation of the problem. In this section, we reduce our Cauchy problem (1.1) to a simpler case and reformulate the problem in terms of the deviation from the asymptotic state (the same as in [20], [28]). At first, without loss of generality, we shall consider the case where $a < 0$, $b = 0$ and the flux function $f(u)$ satisfies

$$\begin{cases} f''(u) > 0 & (u \in (-\infty, a] \cup [0, +\infty)), \\ f(u) = 0 & (u \in (a, 0)), \end{cases} \quad (3.1)$$

under changing the variables and constant as $x - \tilde{\lambda}t \mapsto x$, $u - b \mapsto u$, $f(u+b) - f'(b)u - f(a) \mapsto f(u)$ and $a - b \mapsto a$ in this order. For the far field states $u_{\pm} \in \mathbb{R}$, we only deal with the typical case $a < u_- < 0 < u_+$ for simplicity, since the case $u_- < a < 0 < u_+$ can be treated technically in the same way of the proof as $a < u_- < 0 < u_+$. Indeed, in the case $u_- < a < 0 < u_+$, as we shall see in Section 4 and Section 5, there appears the extra nonlinear interaction terms between two rarefaction waves $u^r(\frac{x}{t}; u_-, a)$ and $u^r(\frac{x}{t}; 0, u_+)$ with $\lambda(a) = \lambda(0) = 0$ in the remainder term of the viscous conservation law for the asymptotics U_{multi} (see the right-hand side of (3.5)). These terms can be handled in much easier way by Lemma 2.2 than that for other essential nonlinear interaction terms between the rarefaction and the viscous contact waves. Furthermore, we should point out that the problem under the assumptions for the flux function (3.1) and the far field states $a < u_- < 0 < u_+$ is essentially the same as that for $a = -\infty$, because obtaining the a priori and the uniform energy estimates for the former one can be given in almost the same way as the latter one. Therefore, it is quite natural for us to treat only a simple case

$$\begin{cases} f''(u) > 0 & (u \in [0, \infty)), \\ f(u) = 0 & (u \in (-\infty, 0)), \end{cases} \quad (3.2)$$

and assume $u_- < 0 < u_+$. The corresponding main theorem is the following.

Theorem 3.1. *Let the flux function $f \in C^1(\mathbb{R}) \cap C^3([0, \infty))$ satisfy (3.2) and the far field states $u_- < 0 < u_+$. Assume that the initial data satisfies $u_0 - u_0^R \in L^2$ and $\partial_x u_0 \in L^{p+1}$. Then the Cauchy problem (1.1) with $p > 1$ has a unique global solution in time u satisfying*

$$\begin{cases} u - u_0^R \in C^0([0, \infty); L^2) \cap L^\infty(\mathbb{R}^+; L^2), \\ \partial_x u \in L^\infty(\mathbb{R}^+; L^{p+1}) \cap L^{p+2}(\mathbb{R}_t^+ \times \{x \in \mathbb{R} \mid u > 0\}), \\ \partial_t u \in L^\infty(\mathbb{R}^+; L^{p+1}), \\ \partial_x(|\partial_x u|^{p-1} \partial_x u) \in L^2(\mathbb{R}_t^+ \times \mathbb{R}_x), \end{cases}$$

and the asymptotic behavior

$$\lim_{t \rightarrow \infty} \sup_{x \in \mathbb{R}} |u(t, x) - U_{multi}(t, x; u_-, u_+)| = 0,$$

where $U_{multi}(t, x) = U_{multi}(t, x; u_-, u_+)$ is defined by

$$U_{multi}(t, x) := U\left(\frac{x}{t^{1/(p+1)}}; u_-, 0\right) + u^r\left(\frac{x}{t}; 0, u_+\right).$$

Here, we first should note by Lemma 2.2 and Lemma 2.3, the asymptotic state $U_{multi}(t, x; u_-, u_+)$ can be replaced by a following approximated one

$$\tilde{U}(t, x) := U(1+t, x) + U^r(t, x) \quad (3.3)$$

where

$$U(1+t, x) = U\left(\frac{x}{(1+t)^{\frac{1}{p+1}}}; u_-, 0\right), \quad U^r(t, x) = U^r(t, x; 0, u_+).$$

This is because, from Lemma 2.2 (especially (4)) and Lemma 2.3 (especially (vi)),

$$\begin{aligned} \sup_{x \in \mathbb{R}} \left| \tilde{U}(t, x) - U_{multi}(t, x) \right| &\leq \sup_{x \in \mathbb{R}} \left| U(1+t, x) - U(t, x) \right| \\ &+ \sup_{x \in \mathbb{R}} \left| U^r(t, x) - u^r\left(\frac{x}{t}\right) \right| \rightarrow 0 \quad (t \rightarrow \infty). \end{aligned}$$

In the following, we write $U(1+t, x)$ again $U(t, x)$ for simplicity. Then it is noted that \tilde{U} approximately satisfies the equation of (1.1) as

$$\partial_t \tilde{U} + \partial_x (f(\tilde{U})) - \mu \partial_x \left(|\partial_x \tilde{U}|^{p-1} \partial_x \tilde{U} \right) = -F_p(U, U^r), \quad (3.4)$$

where the remainder term $F_p(U, U^r)$ is explicitly given by

$$\begin{aligned} F_p(U, U^r) &:= \widetilde{F}_p(U, U^r) \\ &+ \mu \partial_x \left(|\partial_x U + \partial_x U^r|^{p-1} (\partial_x U + \partial_x U^r) - |\partial_x U|^{p-1} \partial_x U \right) \\ &:= - \left(f'(U + U^r) - f'(U^r) \right) \partial_x U^r - f'(U + U^r) \partial_x U \\ &+ \mu \partial_x \left(|\partial_x U + \partial_x U^r|^{p-1} (\partial_x U + \partial_x U^r) - |\partial_x U|^{p-1} \partial_x U \right) \end{aligned} \quad (3.5)$$

which consists of the interaction terms of the viscous contact wave U and the approximation of the rarefaction wave U^r , and the approximation error of U^r as solution to the conservation law for the p -Laplacian type viscosity. Here we should note that U is monotonically nondecreasing and U^r is monotonically increasing, that is, $\partial_x \tilde{U}(t, x) > 0$ ($t \geq 0, x \in \mathbb{R}$) which is frequently used hereinafter. Now putting

$$u(t, x) = \tilde{U}(t, x) + \phi(t, x) \quad (3.6)$$

and using (3.5), we can reformulate the problem (1.1) in terms of the deviation ϕ from \tilde{U} as

$$\begin{cases} \partial_t \phi + \partial_x \left(f(\tilde{U} + \phi) - f(\tilde{U}) \right) \\ \quad - \mu \partial_x \left(|\partial_x \tilde{U} + \partial_x \phi|^{p-1} (\partial_x \tilde{U} + \partial_x \phi) - |\partial_x \tilde{U}|^{p-1} \partial_x \tilde{U} \right) \\ \quad = F_p(U, U^r) \quad (t > 0, x \in \mathbb{R}), \\ \phi(0, x) = \phi_0(x) := u_0(x) - \tilde{U}(0, x) \quad (x \in \mathbb{R}). \end{cases} \quad (3.7)$$

Then we look for the unique global solution in time ϕ which has the asymptotic behavior

$$\sup_{x \in \mathbb{R}} |\phi(t, x)| \xrightarrow{t \rightarrow \infty} 0.$$

Here we note the fact $\phi_0 \in L^2$ and $\partial_x \phi_0 \in L^{p+1}$ by the assumptions on u_0 and the fact

$$\partial_x \tilde{U}(0, \cdot) = \partial_x U(0, \cdot) + \partial_x U^r(0, \cdot) \in L^{p+1}.$$

In the following, we always assume that the flux function $f \in C^1(\mathbb{R}) \cap C^3([0, \infty))$ satisfies (3.2), and the far field states satisfy $u_- < 0 < u_+$. Then the corresponding our main theorem for ϕ we should prove is as follows.

Theorem 3.2. *Suppose $\phi_0 \in L^2$ and $\partial_x \phi_0 \in L^{p+1}$. Then there exists the unique global solution in time $\phi = \phi(t, x)$ of the Cauchy problem (3.7) satisfying*

$$\left\{ \begin{array}{l} \phi \in C^0([0, \infty); L^2) \cap L^\infty(\mathbb{R}^+; L^2), \\ \partial_x \phi \in L^\infty(\mathbb{R}^+; L^{p+1}) \cap L^{p+1}(\mathbb{R}_t^+ \times \mathbb{R}_x), \\ \partial_x(\tilde{U} + \phi) \in L^\infty(\mathbb{R}^+; L^{p+1}) \cap L^{p+2}(\mathbb{R}_t^+ \times \{x \in \mathbb{R} \mid u > 0\}), \\ \partial_t(\tilde{U} + \phi) \in L^\infty(\mathbb{R}^+; L^{p+1}), \\ \partial_x \left(|\partial_x(\tilde{U} + \phi)|^{p-1} \partial_x(\tilde{U} + \phi) \right) \in L^2(\mathbb{R}_t^+ \times \mathbb{R}_x), \end{array} \right.$$

and the asymptotic behavior

$$\lim_{t \rightarrow \infty} \sup_{x \in \mathbb{R}} |\phi(t, x)| = 0.$$

In order to accomplish the proof of Theorem 3.2, we first note that for any $T > 0$, the global existence on $[0, T]$ and uniqueness can be proved by the similar arguments as in [18]. Indeed, we rewrite our Cauchy problem (3.7) again as

$$\left\{ \begin{array}{ll} \partial_t \phi + \partial_x (f(U + U^r + \phi) - f(U^r)) + \mu \partial_x ((\partial_x U)^p) \\ \quad = \mu \partial_x \left(|\partial_x(U + U^r + \phi)|^{p-1} \partial_x(U + U^r + \phi) \right) & (t > 0, x \in \mathbb{R}), \\ \phi(0, x) = \phi_0(x) := u_0(x) - \tilde{U}(0, x) & (x \in \mathbb{R}), \end{array} \right. \quad (3.8)$$

and for any $\epsilon \in (0, 1]$, we consider the ϵ -regularized problem as

$$\left\{ \begin{array}{ll} \partial_t \phi_\epsilon + \partial_x \left(f^\epsilon(U^\epsilon + U^{r, \epsilon} + \phi_\epsilon) - f^\epsilon(U^{r, \epsilon}) \right) + \mu \partial_x ((\partial_x U^\epsilon)^p) \\ \quad = \mu \partial_x \left(\left((\partial_x(U^\epsilon + U^{r, \epsilon} + \phi_\epsilon))^2 + \epsilon \right)^{\frac{p-1}{2}} \partial_x(U^\epsilon + U^{r, \epsilon} + \phi_\epsilon) \right) & (t > 0, x \in \mathbb{R}), \\ \phi_\epsilon(0, x) = \phi_0^\epsilon(x) & (x \in \mathbb{R}). \end{array} \right. \quad (3.9)$$

where,

$$\left\{ \begin{array}{l} \phi_0^\epsilon(x) := (\rho_\epsilon *_x \phi_0)(x) := \int_{-\infty}^{\infty} \rho_\epsilon(x-y) \phi_0(y) dy \in H^\infty(\mathbb{R}_x), \\ f^\epsilon(u) := (\rho_\epsilon *_u f)(u) := \int_{-\infty}^{\infty} \rho_\epsilon(u-v) f(v) dv \in C^\infty(\mathbb{R}_u), \\ U^{r,\epsilon}(t, x) := \left((f^\epsilon)' \right)^{-1} \left(w(t, x; f^\epsilon(0), f^\epsilon(w_+)) \right) \in \mathcal{B}^\infty([0, \infty)_t \times \mathbb{R}_x), \\ U^\epsilon(t, x) := (\rho_\epsilon *_x U)(t, x) := \int_{-\infty}^{\infty} \rho_\epsilon(x-y) U \left(\frac{y}{(1+t)^{\frac{1}{p+1}}} \right) dy \\ \qquad \qquad \qquad \in \mathcal{B}^1 \cap C^\infty((-1, \infty)_t \times \mathbb{R}_x). \end{array} \right.$$

Here, $w(t, x)$ is the classical solution of (2.4) with $w_- = 0$. If we define $u_\epsilon := \phi_\epsilon + U^\epsilon + U^{r,\epsilon}$, we also have the equivalent form

$$\left\{ \begin{array}{l} \partial_t u_\epsilon + \partial_x \left(f^\epsilon(u_\epsilon) \right) = \mu \partial_x \left(\left((\partial_x u_\epsilon)^2 + \epsilon \right)^{\frac{p-1}{2}} \partial_x u_\epsilon \right) \\ \qquad \qquad \qquad (t > 0, x \in \mathbb{R}), \\ u_\epsilon(0, x) = u_0^\epsilon(x) := \phi_0^\epsilon(x) + U^\epsilon(0, x) + U^{r,\epsilon}(0, x) \quad (x \in \mathbb{R}). \end{array} \right. \quad (3.10)$$

Owing to Ladyženskaja-Solonnikov-Ural'ceva [12] (see also [15]), the regularized problem (3.9) has a unique classical global solution in time $\phi_\epsilon = \phi_\epsilon(t, x)$ on $[0, T] \times \mathbb{R}$ for any $T > 0$, that is,

$$\phi_\epsilon, \partial_x \phi_\epsilon, \partial_x^2 \phi_\epsilon, \partial_t \phi_\epsilon \in C^\infty([0, \infty) \times \mathbb{R})$$

because the equation in (3.9) is uniformly parabolic with variable coefficients. Further, the maximum principles (see [9], [26]) for (3.10) allows us to get the uniform boundedness to $\phi_\epsilon(t, x)$ as follows.

Lemma 3.1 (uniform boundedness). *It holds that*

$$\begin{aligned} & \sup_{t \in [0, \infty), x \in \mathbb{R}} |\phi_\epsilon(t, x)| \\ & \leq \sup_{x \in \mathbb{R}} |u_0^\epsilon(x)| + \sup_{t \in [0, \infty), x \in \mathbb{R}} |U(t, x)| + \sup_{t \in [0, \infty), x \in \mathbb{R}} |U^r(t, x)| \\ & = \|\phi_0\|_{L^\infty} + 2|u_-| + 2|u_+| =: \tilde{C}. \end{aligned}$$

Since $\phi_0^\epsilon \in H^\infty$, by using the above uniform boundedness and the standard arguments in the Sobolev space on the uniformly parabolic equations, we can see the classical C^∞ -solution ϕ_ϵ also satisfies

$$\phi_\epsilon \in C^\infty([0, \infty); H^\infty).$$

Then, for any fixed $\epsilon \in (0, 1)$ and $T > 0$, we can obtain the following a priori estimates which are depend upon ϵ and T to the problem (3.9) as follows.

Lemma 3.2 (a priori estimates I). *There exists a positive constant C_I such that for $0 < t < T$,*

$$\begin{aligned} & \|\phi_\epsilon(t)\|_{L^2}^2 + \int_0^t \int_{-\infty}^{\infty} \phi^2 (\partial_x U^\epsilon + \partial_x U^{r,\epsilon}) \, dx d\tau \\ & + \int_0^t \int_{-\infty}^{\infty} (\partial_x u_\epsilon)^2 \left((\partial_x u_\epsilon)^2 + \epsilon \right)^{\frac{p-1}{2}} \, dx d\tau \leq C_I, \end{aligned}$$

where

$$C_I = C\left(T, \tilde{C}, \|\phi_0\|_{L^2}\right).$$

Lemma 3.3 (a priori estimates II). *There exists a positive constant C_{II} such that for $0 < t < T$,*

$$\begin{aligned} & \|\partial_x u_\epsilon(t)\|_{L^{p+1}}^{p+1} + \epsilon^{\frac{p-1}{2}} \|\partial_x u_\epsilon(t)\|_{L^2}^2 \\ & + \int_0^t \int_{-\infty}^{\infty} \left((\partial_x u_\epsilon)^2 + \epsilon \right)^{p-1} (\partial_x^2 u_\epsilon)^2 \, dx d\tau \\ & + \int_0^t \int_{-\infty}^{\infty} \left((\partial_x u_\epsilon)^2 + \epsilon \right)^{\frac{p-1}{2}} |\partial_x u_\epsilon|^3 \, dx d\tau \leq C_{II}, \end{aligned}$$

where

$$C_{II} = C\left(T, \tilde{C}, \|\phi_0\|_{L^2}, \|\partial_x u_0\|_{L^{p+1}}, \epsilon^{\frac{p-1}{2}} \|\partial_x u_0\|_{L^2}\right).$$

The proofs of Lemma 3.2 and Lemma 3.3 are given in the last part in this section. We can also prove the following lemma. Because the proof is given in the same way as the above lemma, we omit the proof.

Lemma 3.4 (a priori estimates III). *There exists a positive constant C_{III} such that for $0 < t < T$,*

$$\|\partial_x u_\epsilon(t)\|_{L^2} \leq C_{III}$$

where

$$C_{III} = C\left(T, \tilde{C}, \|\phi_0\|_{L^2}, \|\partial_x u_0\|_{L^{p+1}}, \|\partial_x u_0\|_{L^2}\right).$$

Once the Lemmas 3.2, 3.3 and 3.4 are proved, taking the limit $\epsilon \searrow 0$ as in the arguments in [18], we have Theorem 3.3.

Theorem 3.3. *For any initial data $\phi_0 \in L^2$ and $\partial_x \phi_0 \in L^{p+1}$, there exists the unique global solution in time of the Cauchy problem (3.8) $\phi = \phi(t, x)$ satisfying for any $T > 0$,*

$$\begin{cases} \phi \in C^0([0, T]; L^2), \\ \partial_x \phi \in L^\infty(0, T; L^{p+1}), \\ \partial_x \left(|\partial_x (U + U^r + \phi)|^{p-1} \partial_x (U + U^r + \phi) \right) \in L^2(0, T; L^2). \end{cases}$$

Furthermore, the solution satisfies the uniform boundedness

$$\sup_{t \in [0, \infty), x \in \mathbb{R}} |\phi(t, x)| \leq \tilde{C} \left(= \|\phi_0\|_{L^\infty} + 2|u_-| + 2|u_+| \right),$$

and also satisfies for any $T > 0$,

$$\begin{aligned} \|\phi(t)\|_{L^2}^2 + \int_0^t \int_{-\infty}^{\infty} \phi^2 (\partial_x U + \partial_x U^r) dx d\tau \\ + \int_0^t \int_{-\infty}^{\infty} |\partial_x u|^{p+1} dx d\tau \leq \tilde{C}_I \quad (t \in [0, T]), \end{aligned}$$

where

$$\tilde{C}_I = C \left(T, \tilde{C}, \|\phi_0\|_{L^2} \right),$$

and

$$\begin{aligned} \|\partial_x u(t)\|_{L^{p+1}}^{p+1} + \int_0^t \int_{-\infty}^{\infty} |\partial_x u|^{2(p-1)} (\partial_x^2 u)^2 dx d\tau \\ + \int_0^t \int_{-\infty}^{\infty} |\partial_x u|^{p+2} dx d\tau \leq \tilde{C}_{II} \quad (t \in [0, T]), \end{aligned}$$

where

$$\tilde{C}_{II} = C \left(T, \tilde{C}, \|\phi_0\|_{L^2}, \|\partial_x u_0\|_{L^{p+1}} \right).$$

Indeed, for the initial data which satisfies

$$\phi_0 \in L^2, \quad \partial_x \phi_0 \in L^2 \cap L^{p+1},$$

we can take a subsequence of $\{u_\epsilon\}$ (write $\{u_\epsilon\}$ again for simplicity) and a limit function ϕ (correspondingly $u := \phi + U + U^r$) by Lemmas 3.2, 3.3 and 3.4, such that

$$\left\{ \begin{array}{l} \text{s-}\lim_{\epsilon \rightarrow 0} \phi_\epsilon = \phi \in C^0([0, T]; L^2), \\ \text{w}^* \text{-}\lim_{\epsilon \rightarrow 0} \partial_x \phi_\epsilon = \partial_x \phi \in L^\infty(0, T; L^{p+1}), \\ \text{w}^* \text{-}\lim_{\epsilon \rightarrow 0} \partial_x u_\epsilon = \partial_x u \in L^\infty(0, T; L^{p+1}), \\ \text{w-}\lim_{\epsilon \rightarrow 0} \partial_x \left(|\partial_x u_\epsilon|^{p-1} \partial_x u_\epsilon \right) = \partial_x \left(|\partial_x u|^{p-1} \partial_x u \right) \in L^2(0, T; L^2), \\ \text{w-}\lim_{\epsilon \rightarrow 0} \partial_x \left(\left((\partial_x u_\epsilon)^2 + \epsilon \right)^{\frac{p-1}{2}} \partial_x u_\epsilon \right) = \partial_x \left(|\partial_x u|^{p-1} \partial_x u \right) \in L^2(0, T; L^2). \end{array} \right.$$

We can also see that the limit function ϕ gives the unique global solution of (3.8) and the results in Theorem 3.3 hold. In particular, we note that the energy estimates in Theorem 3.3 are independent of $\|\partial_x \phi_0\|_{L^2}$. For the initial data which satisfies

$$\phi_0 \in L^2, \quad \partial_x \phi_0 \in L^{p+1},$$

we take again the approximate sequence $\{\phi_0^\delta\}$ which satisfies

$$\phi_0^\delta \in L^2, \quad \partial_x \phi_0^\delta \in L^2 \cap L^{p+1}$$

and

$$\begin{cases} \text{s-}\lim_{\delta \rightarrow 0} \phi_0^\delta = \phi_0 \in L^2, \\ \text{s-}\lim_{\delta \rightarrow 0} \partial_x \phi_0^\delta = \partial_x \phi_0 \in L^{p+1}. \end{cases}$$

We may take the limit $\delta \rightarrow 0$ to get Theorem 3.3.

Since the energy estimates in Theorem 3.3 depend on T , we can not prove the asymptotics

$$\|\phi(t)\|_{L^\infty} \rightarrow 0 \quad (t \rightarrow \infty).$$

In order to show the desired asymptotics, we show the following a priori estimates which are independent of T in the next sections.

Proposition 3.1 (uniform estimates I). *For any initial data $\phi_0 \in L^2$ and $\partial_x \phi_0 \in L^{p+1}$, there exists a positive constant*

$$C_p(\phi_0) = C(\|\phi_0\|_{L^2})$$

such that the unique global solution in time ϕ to the Cauchy problem (3.8) constructed in Theorem 3.3 satisfies

$$\begin{aligned} & \|\phi(t)\|_{L^2}^2 + \int_0^t G(\tau) d\tau \\ & + \int_0^t \int_{-\infty}^{\infty} (\partial_x \phi)^2 \left(|\partial_x \phi|^{p-1} + |\partial_x U|^{p-1} + |\partial_x U^r|^{p-1} \right) dx d\tau \leq C_p(\phi_0) \end{aligned} \quad (3.11)$$

for $t \geq 0$, where

$$\begin{aligned} G(t) := & \left(\int_{\tilde{U} \geq 0} \phi^2 \partial_x \tilde{U} dx \right) (t) + \left(\int_{\tilde{U} + \phi \geq 0, \tilde{U} < 0} (\tilde{U} + \phi)^2 \partial_x \tilde{U} dx \right) (t) \\ & + \left(\int_{\tilde{U} + \phi < 0, \tilde{U} \geq 0} (\tilde{U} + |\phi|)^2 \partial_x \tilde{U} dx \right) (t). \end{aligned}$$

Furthermore, we have the L^{p+1} -energy estimate for $\partial_x u$ as follows.

Proposition 3.2 (uniform estimates II). *For any initial data $\phi_0 \in L^2$ and $\partial_x \phi_0 \in L^{p+1}$, there exists a positive constant*

$$C_p(\phi_0, \partial_x u_0) = C(\|\phi_0\|_{L^2}, \|\partial_x u_0\|_{L^{p+1}})$$

such that for $t \geq 0$,

$$\begin{aligned} & \|\partial_x u(t)\|_{L^{p+1}}^{p+1} + \int_0^t \int_{-\infty}^{\infty} |\partial_x u|^{2(p-1)} (\partial_x^2 u)^2 dx d\tau \\ & + \int_0^t \|\partial_x u(\tau)\|_{L^{p+2}(\{x \in \mathbb{R} \mid u > 0\})}^{p+2} d\tau \leq C_p(\phi_0, \partial_x u_0). \end{aligned} \quad (3.12)$$

Proof of Lemma 3.2. We first note

$$\begin{cases} \|\phi_0^\epsilon\|_{L^q} \leq \|\phi_0\|_{L^q} & (1 \leq q \leq \infty), \\ \|\partial_x u_0^\epsilon\|_{L^{r+1}} \leq \|\partial_x u_0\|_{L^{r+1}} & (r \geq 1), \\ \sup_{-\tilde{C} \leq u \leq \tilde{C}} |D^k f^\epsilon(u)| \leq \sup_{0 \leq u \leq \tilde{C}+1} |D^k f(u)| & (k = 0, 1, 2), \end{cases} \quad (3.13)$$

where the positive constant \tilde{C} is defined in Lemma 3.1. By using Lemma 1.2.1, we can get

Lemma 3.5.

(1) $U^{r,\epsilon}(t, x)$ is the unique C^∞ -global solution in space-time of the Cauchy problem

$$\begin{cases} \partial_t U^{r,\epsilon} + \partial_x (f^\epsilon(U^{r,\epsilon})) = 0 & (t > 0, x \in \mathbb{R}), \\ U^{r,\epsilon}(0, x) = \left((f^\epsilon)' \right)^{-1} \left(\frac{(f^\epsilon)'(w_+)}{2} + \frac{(f^\epsilon)'(w_+)}{2} \tanh x \right) & (x \in \mathbb{R}), \\ \lim_{x \rightarrow +\infty} U^{r,\epsilon}(t, x) = (f^\epsilon)'(w_+) & (t \geq 0), \\ \lim_{x \rightarrow -\infty} U^{r,\epsilon}(t, x) = (f^\epsilon)'(0) & (t \geq 0). \end{cases}$$

(2) $(f^\epsilon)'(0) \leq U^{r,\epsilon}(t, x) \leq (f^\epsilon)'(w_+)$ and $\partial_x U^{r,\epsilon}(t, x) \geq 0$ $(t > 0, x \in \mathbb{R})$.

(3) For any $1 \leq q \leq \infty$, there exists a positive constant C_q such that

$$\begin{aligned} \|\partial_x U^{r,\epsilon}(t)\|_{L^q} &\leq C(q, \tilde{C}) (1+t)^{-1+\frac{1}{q}} \quad (t \geq 0), \\ \|\partial_x^2 U^{r,\epsilon}(t)\|_{L^q} &\leq C(q, \tilde{C}) (1+t)^{-1} \quad (t \geq 0). \end{aligned}$$

(4) It holds that

$$\lim_{t \rightarrow \infty} \sup_{x \in \mathbb{R}} \left| U^{r,\epsilon}(t, x) - \left((f^\epsilon)' \right)^{-1} \left(w^r \left(\frac{x}{t}; f^\epsilon(0), f^\epsilon(w_+) \right) \right) \right| = 0,$$

where w^r is the solution of (1.2.1) with $w_- = 0$.

On the other hand, we easily have

$$\begin{aligned} |U^\epsilon(t, x)| &= \int_{-\infty}^{\infty} \rho_\epsilon(y) |U(t, x-y)| dy \\ &\leq \int_{-\infty}^{\infty} \rho_\epsilon(y) dy \|U(t)\|_{L^\infty} \leq |u_-|, \end{aligned}$$

and

$$\begin{aligned} \partial_x U^\epsilon(t, x) &= \int_{-\infty}^{\infty} \rho_\epsilon(y) \partial_x U(t, x-y) dy \\ &\leq \int_{-\infty}^{\infty} \rho_\epsilon(y) dy \|\partial_x U(t)\|_{L^\infty}, \end{aligned} \quad (3.14)$$

then we easily have by Lemma 2.3,

$$\begin{aligned} \|\partial_x U^\epsilon(t)\|_{L^\infty} &\leq \|\partial_x U(t)\|_{L^\infty} \\ &\leq (2A)^{\frac{1}{p-1}} (1+t)^{-\frac{1}{p+1}} \leq (2A)^{\frac{1}{p-1}}. \end{aligned} \quad (3.15)$$

We also have for $q \geq 1$,

$$\begin{aligned} \int_{-\infty}^{\infty} (\partial_x U^\epsilon)^q dx &\leq \|\partial_x U^\epsilon(t)\|_{L^\infty}^{q-1} \int_{-\infty}^{\infty} \rho_\epsilon(y) \left[U(t, x-y) \right]_{x \rightarrow -\infty}^{x \rightarrow +\infty} dy \\ &\leq (2A)^{\frac{q-1}{p-1}} |u_-| (1+t)^{-\frac{q-1}{p+1}}, \end{aligned} \quad (3.16)$$

then we get

$$\begin{aligned} \|\partial_x U^\epsilon(t)\|_{L^q} &\leq (2A)^{\frac{q-1}{(p-1)q}} |u_-|^{\frac{1}{q}} (1+t)^{-\frac{q-1}{(p+1)q}} \\ &\sim \|\partial_x U(t)\|_{L^q}. \end{aligned} \quad (3.17)$$

Multiplying the equation in (3.9) by ϕ_ϵ , we obtain the divergence form

$$\begin{aligned} &\partial_t \left(\frac{1}{2} |\phi_\epsilon|^2 \right) + \partial_x \left(\phi_\epsilon \left(f^\epsilon(U^\epsilon + U^{r,\epsilon} + \phi_\epsilon) - f^\epsilon(U^\epsilon + U^{r,\epsilon}) \right) \right) \\ &+ \partial_x \left(- \int_{U^\epsilon + U^{r,\epsilon}}^{U^\epsilon + U^{r,\epsilon} + \phi_\epsilon} f^\epsilon(s) ds + f^\epsilon(U^\epsilon + U^{r,\epsilon}) \phi_\epsilon \right) \\ &+ \partial_x \left(-\mu \phi_\epsilon \left(\left((\partial_x u_\epsilon)^2 + \epsilon \right)^{\frac{p-1}{2}} \partial_x u_\epsilon - (\partial_x U^\epsilon)^p \right) \right) \\ &+ \left(f^\epsilon(U^\epsilon + U^{r,\epsilon} + \phi_\epsilon) - f^\epsilon(U^\epsilon + U^{r,\epsilon}) - (f^\epsilon)'(U^\epsilon + U^{r,\epsilon}) \phi_\epsilon \right) \\ &\quad \times (\partial_x U^\epsilon + \partial_x U^{r,\epsilon}) \\ &+ \mu (\partial_x \phi_\epsilon) \left(\left((\partial_x u_\epsilon)^2 + \epsilon \right)^{\frac{p-1}{2}} \partial_x u_\epsilon \right) \\ &= -\phi_\epsilon \partial_x (f^\epsilon(U^\epsilon + U^{r,\epsilon}) - f^\epsilon(U^{r,\epsilon})) + \mu (\partial_x \phi_\epsilon) ((\partial_x U^\epsilon)^p). \end{aligned} \quad (3.18)$$

Integrating (3.18) with respect to x and t , we have

$$\begin{aligned} &\frac{1}{2} \|\phi_\epsilon(t)\|_{L^2}^2 \\ &+ \int_0^t \int_{-\infty}^{\infty} \left(f^\epsilon(U^\epsilon + U^{r,\epsilon} + \phi_\epsilon) - f^\epsilon(U^\epsilon + U^{r,\epsilon}) - (f^\epsilon)'(U^\epsilon + U^{r,\epsilon}) \phi_\epsilon \right) \\ &\quad \times (\partial_x U^\epsilon + \partial_x U^{r,\epsilon}) dx d\tau \\ &+ \mu \int_0^t \int_{-\infty}^{\infty} (\partial_x \phi_\epsilon) \left(\left((\partial_x u_\epsilon)^2 + \epsilon \right)^{\frac{p-1}{2}} \partial_x u_\epsilon \right) dx d\tau \\ &= \frac{1}{2} \|\phi_0\|_{L^2}^2 + \int_0^t \int_{-\infty}^{\infty} -\phi_\epsilon \left((f^\epsilon)'(U^\epsilon + U^{r,\epsilon}) - (f^\epsilon)'(U^{r,\epsilon}) \right) (\partial_x U^{r,\epsilon}) dx d\tau \\ &+ \int_0^t \int_{-\infty}^{\infty} -\phi_\epsilon \left((f^\epsilon)'(U^\epsilon + U^{r,\epsilon}) \right) (\partial_x U^\epsilon) dx d\tau \\ &+ \mu \int_0^t \int_{-\infty}^{\infty} (\partial_x \phi_\epsilon) ((\partial_x U^\epsilon)^p) dx d\tau. \end{aligned} \quad (3.19)$$

Since the second term on the left-hand side of (3.19) is equal to

$$\begin{aligned} & \frac{1}{2} \int_0^t \int_{-\infty}^{\infty} (f^\epsilon)'' (\theta \phi_\epsilon + U^\epsilon + U^{r,\epsilon}) \phi_\epsilon^2 (\partial_x U^\epsilon + \partial_x U^{r,\epsilon}) dx d\tau \\ & (\exists \theta = \theta(t, x) \in (0, 1)), \end{aligned} \quad (3.20)$$

and since $\partial_x U^\epsilon + \partial_x U^r \geq 0$ by Lemma 2.2 and Lemma 2.3, the term is nonnegative. The third term on the left-hand side is also equal to

$$\mu \int_0^t \int_{-\infty}^{\infty} \left(\partial_x u_\epsilon - (\partial_x U^\epsilon + \partial_x U^{r,\epsilon}) \right) \left(((\partial_x u_\epsilon)^2 + \epsilon)^{\frac{p-1}{2}} \partial_x u_\epsilon \right) dx d\tau \quad (3.21)$$

and we can estimate it as

$$\begin{aligned} & \mu \int_0^t \int_{-\infty}^{\infty} \left((\partial_x u_\epsilon)^2 + \epsilon \right)^{\frac{p-1}{2}} \partial_x u_\epsilon (\partial_x U^\epsilon + \partial_x U^{r,\epsilon}) dx d\tau \\ & \leq \frac{\mu}{2} \int_0^t \int_{-\infty}^{\infty} \left((\partial_x u_\epsilon)^2 + \epsilon \right)^{\frac{p-1}{2}} \left((\partial_x u_\epsilon)^2 + (\partial_x U^\epsilon + \partial_x U^{r,\epsilon})^2 \right) dx d\tau \\ & \leq \frac{\mu}{2} \int_0^t \int_{-\infty}^{\infty} \left((\partial_x u_\epsilon)^2 + \epsilon \right)^{\frac{p-1}{2}} (\partial_x u_\epsilon)^2 dx d\tau \\ & \quad + \int_0^t \int_{-\infty}^{\infty} \frac{\mu}{2} 2^{\frac{p-1}{2}} \left(|\partial_x u_\epsilon|^{p-1} + \epsilon^{\frac{p-1}{2}} \right) (\partial_x U^\epsilon + \partial_x U^{r,\epsilon})^2 dx d\tau \\ & \leq \frac{3}{4} \mu \int_0^t \int_{-\infty}^{\infty} \left((\partial_x u_\epsilon)^2 + \epsilon \right)^{\frac{p-1}{2}} (\partial_x u_\epsilon)^2 dx d\tau \\ & \quad + \frac{\mu}{2} 2^{\frac{p-1}{2}} \epsilon^{\frac{p-1}{2}} \int_0^t \int_{-\infty}^{\infty} (\partial_x U^\epsilon + \partial_x U^{r,\epsilon})^2 dx d\tau \\ & \quad + C_\mu \int_0^t \int_{-\infty}^{\infty} (\partial_x U^\epsilon + \partial_x U^{r,\epsilon})^{p+1} dx d\tau \\ & \leq \frac{3}{4} \mu \int_0^t \int_{-\infty}^{\infty} \left((\partial_x u_\epsilon)^2 + \epsilon \right)^{\frac{p-1}{2}} (\partial_x u_\epsilon)^2 dx d\tau \\ & \quad + C_{p,\mu,u_\pm} \epsilon^{\frac{p-1}{2}} T + C_{p,\mu,u_\pm} T. \end{aligned} \quad (3.22)$$

Substituting (3.20) and (3.21) with (3.22) into (3.19), we have

$$\begin{aligned} & \frac{1}{2} \|\phi_\epsilon(t)\|_{L^2}^2 + \frac{1}{2} C_{p,\epsilon}^{-1} \int_0^t \int_{-\infty}^{\infty} \phi_\epsilon^2 (\partial_x U^\epsilon + \partial_x U^{r,\epsilon}) dx d\tau \\ & \quad + \frac{\mu}{4} \int_0^t \int_{-\infty}^{\infty} (\partial_x u_\epsilon)^2 \left((\partial_x u_\epsilon)^2 + \epsilon \right)^{\frac{p-1}{2}} dx d\tau \\ & \leq \frac{1}{2} \|\phi_0^\epsilon\|_{L^2}^2 + C_{p,\mu,u_\pm} \epsilon^{\frac{p-1}{2}} T + C_{p,\mu,u_\pm} T \\ & \quad + \left| \int_0^t \int_{-\infty}^{\infty} -\phi_\epsilon \left((f^\epsilon)'(U^\epsilon + U^{r,\epsilon}) - (f^\epsilon)'(U^{r,\epsilon}) \right) (\partial_x U^{r,\epsilon}) dx d\tau \right| \\ & \quad + \left| \int_0^t \int_{-\infty}^{\infty} -\phi_\epsilon \left((f^\epsilon)'(U^\epsilon + U^{r,\epsilon}) \right) (\partial_x U^\epsilon) dx d\tau \right| \\ & \quad + \mu \left| \int_0^t \int_{-\infty}^{\infty} (\partial_x \phi_\epsilon) ((\partial_x U^\epsilon)^p) dx d\tau \right|. \end{aligned} \quad (3.23)$$

Next, by using Lemma 2.2, Lemma 3.1 and (3.15), we estimate the second term on the right-hand side of (3.23) as

$$\begin{aligned}
& \left| \int_0^t \int_{-\infty}^{\infty} -\phi_{\epsilon} \left((f^{\epsilon})'(U^{\epsilon} + U^{r,\epsilon}) - (f^{\epsilon})'(U^{r,\epsilon}) \right) (\partial_x U^{r,\epsilon}) dx d\tau \right| \\
& \leq \sup_{t \in [0, \infty), x \in \mathbb{R}} |\phi_{\epsilon}(t, x)| \sup_{-\tilde{C} \leq u \leq \tilde{C}} \left| (f^{\epsilon})''(u) \right| \\
& \quad \times \int_0^t \int_{-\infty}^{\infty} |U^{\epsilon}| |\partial_x U^r| dx d\tau \\
& \leq C^{\dagger} T,
\end{aligned} \tag{3.24}$$

where

$$C^{\dagger} = C \left(\tilde{C}, \sup_{-\tilde{C} \leq u \leq \tilde{C}} \left| (f^{\epsilon})''(u) \right| \right).$$

Similarly, we also estimate the third term on the right-hand side as

$$\begin{aligned}
& \left| \int_0^t \int_{-\infty}^{\infty} -\phi_{\epsilon} \left((f^{\epsilon})'(U^{\epsilon} + U^{r,\epsilon}) \right) (\partial_x U^{\epsilon}) dx d\tau \right| \\
& \leq \sup_{t \in [0, \infty), x \in \mathbb{R}} |\phi_{\epsilon}(t, x)| \sup_{-\tilde{C} \leq u \leq \tilde{C}} \left| (f^{\epsilon})'(u) \right| \\
& \quad \times \int_0^t \int_{-\infty}^{\infty} |U^{\epsilon} + U^{r,\epsilon}| |\partial_x U^{\epsilon}| dx d\tau \\
& \leq C^{\dagger} T.
\end{aligned} \tag{3.25}$$

Finally, we estimate the fourth term on the right-hand side as

$$\begin{aligned}
& \mu \left| \int_0^t \int_{-\infty}^{\infty} (\partial_x \phi_{\epsilon}) ((\partial_x U^{\epsilon})^p) dx d\tau \right| \\
& \leq \int_0^t \int_{-\infty}^{\infty} \left(\frac{\mu}{8} |\partial_x u_{\epsilon}|^{p+1} + C_{\mu} |\partial_x U^{\epsilon}|^{p+1} \right) dx d\tau \\
& \leq \frac{\mu}{8} \int_0^t \int_{-\infty}^{\infty} \left((\partial_x u_{\epsilon})^2 + \epsilon \right)^{\frac{p-1}{2}} (\partial_x u_{\epsilon})^2 dx d\tau + C_{\mu} \int_0^t \|\partial_x U^{\epsilon}(\tau)\|_{L^{p+1}}^{p+1} d\tau.
\end{aligned} \tag{3.26}$$

Substituting (3.23), (3.24) and (3.25) with (3.16) into (3.22), we get the desired a priori energy inequality

$$\begin{aligned}
& \frac{1}{2} \|\phi_{\epsilon}(t)\|_{L^2}^2 + \frac{1}{2} C_{p,\epsilon}^{-1} \int_0^t \int_{-\infty}^{\infty} \phi_{\epsilon}^2 (\partial_x U^{\epsilon} + \partial_x U^{r,\epsilon}) dx d\tau \\
& \quad + \frac{\mu}{8} \int_0^t \int_{-\infty}^{\infty} (\partial_x u_{\epsilon})^2 \left((\partial_x u_{\epsilon})^2 + \epsilon \right)^{\frac{p-1}{2}} dx d\tau \\
& \leq \frac{1}{2} \|\phi_0\|_{L^2}^2 + C^{\dagger\dagger} T,
\end{aligned} \tag{3.27}$$

where

$$C^{\dagger\dagger} = C \left(\tilde{C}, \sup_{-\tilde{C} \leq u \leq \tilde{C}} \left| (f^{\epsilon})'(u) \right|, \sup_{-\tilde{C} \leq u \leq \tilde{C}} \left| (f^{\epsilon})''(u) \right| \right).$$

Thus, by noting (3.13), we complete the proof of Lemma 3.2.

Proof of Lemma 3.3. Multiplying the equation in (3.10) by

$$-\partial_x \left(\left((\partial_x u_\epsilon)^2 + \epsilon \right)^{\frac{p-1}{2}} \partial_x u_\epsilon \right),$$

we have the following divergence form

$$\begin{aligned} & \partial_t \left(\int_0^{\partial_x u_\epsilon} (s^2 + \epsilon)^{\frac{p-1}{2}} s \, ds \right) \\ & + \partial_x \left(- \left((\partial_x u_\epsilon)^2 + \epsilon \right)^{\frac{p-1}{2}} \partial_x u_\epsilon \left(\partial_t u_\epsilon + \partial_x (f^\epsilon(u_\epsilon)) \right) \right) \\ & + \partial_x \left((f^\epsilon)'(u_\epsilon) \int_0^{\partial_x u_\epsilon} (s^2 + \epsilon)^{\frac{p-1}{2}} s \, ds \right) \\ & + \mu \left(\partial_x \left(\left((\partial_x u_\epsilon)^2 + \epsilon \right)^{\frac{p-1}{2}} \partial_x u_\epsilon \right) \right)^2 \\ & = - (f^\epsilon)''(u_\epsilon) \left((\partial_x u_\epsilon)^2 + \epsilon \right)^{\frac{p-1}{2}} (\partial_x u_\epsilon)^3 \\ & \quad + (f^\epsilon)''(u_\epsilon) \partial_x u_\epsilon \int_0^{\partial_x u_\epsilon} (s^2 + \epsilon)^{\frac{p-1}{2}} s \, ds. \end{aligned} \tag{3.28}$$

Integrating (3.28) with respect to t and x , we have

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_0^{\partial_x u_\epsilon(t)} (s^2 + \epsilon)^{\frac{p-1}{2}} s \, ds dx \\ & + \mu \int_0^t \int_{-\infty}^{\infty} \left(\partial_x \left(\left((\partial_x u_\epsilon)^2 + \epsilon \right)^{\frac{p-1}{2}} \partial_x u_\epsilon \right) \right)^2 dx d\tau \\ & \leq \int_{-\infty}^{\infty} \int_0^{\partial_x u_0^\epsilon} (s^2 + \epsilon)^{\frac{p-1}{2}} s \, ds dx \\ & \quad + \int_0^t \int_{-\infty}^{\infty} \left| (f^\epsilon)''(u_\epsilon) \right| \left((\partial_x u_\epsilon)^2 + \epsilon \right)^{\frac{p-1}{2}} |\partial_x u_\epsilon|^3 dx d\tau \\ & \quad + \int_0^t \int_{-\infty}^{\infty} \left| (f^\epsilon)''(u_\epsilon) \right| |\partial_x u_\epsilon| \left| \int_0^{\partial_x u_\epsilon} (s^2 + \epsilon)^{\frac{p-1}{2}} s \, ds \right| dx d\tau \\ & \leq \int_{-\infty}^{\infty} \int_0^{\partial_x u_0^\epsilon} (s^2 + \epsilon)^{\frac{p-1}{2}} s \, ds dx \\ & \quad + 2 \sup_{-\tilde{C} \leq u \leq \tilde{C}} \left| (f^\epsilon)''(u) \right| \int_0^t \int_{-\infty}^{\infty} \left((\partial_x u_\epsilon)^2 + \epsilon \right)^{\frac{p-1}{2}} |\partial_x u_\epsilon|^3 dx d\tau. \end{aligned} \tag{3.29}$$

In order to estimate the second term on the right-hand side of above energy inequality, we use the following lemma. Since the proof is elementary we omit the proof.

Lemma 3.6. *There exists a positive constant $C_{p,q}$ such that for $v \in L^{2(q-p+1)}$ with*

$\partial_x v \in L^2$ ($p > 1$, $q > p - 1$), it holds

$$\begin{aligned} \sup_{x \in \mathbb{R}} |v(x)| &\leq C_{p,q} \left(\int_{-\infty}^{\infty} |v(x)|^{2(q-p+1)} dx \right)^{\frac{1}{2(q+1)}} \\ &\quad \times \left(\int_{-\infty}^{\infty} |v(x)|^{2(p-1)} (\partial_x v(x))^2 dx \right)^{\frac{1}{2(q+1)}}. \end{aligned}$$

By using Lemma 3.6 with $v = \partial_x u_\epsilon$ and $q = \frac{3}{2}p$, we can estimate the second term as

$$\begin{aligned} &\int_{-\infty}^{\infty} \left((\partial_x u_\epsilon)^2 + \epsilon \right)^{\frac{p-1}{2}} |\partial_x u_\epsilon|^3 dx \\ &\leq C_p \left(\int_{-\infty}^{\infty} |\partial_x u_\epsilon|^{p+2} dx \right)^{\frac{2}{3p+2}} \left(\int_{-\infty}^{\infty} |\partial_x u_\epsilon|^{2(p-1)} (\partial_x^2 u_\epsilon)^2 dx \right)^{\frac{2}{3p+2}} \\ &\quad \times \int_{-\infty}^{\infty} \left((\partial_x u_\epsilon)^2 + \epsilon \right)^{\frac{p-1}{2}} (\partial_x u_\epsilon)^2 dx. \end{aligned} \quad (3.30)$$

Noting

$$\int_{-\infty}^{\infty} |\partial_x u_\epsilon|^{p+2} dx \leq C_p \int_{-\infty}^{\infty} \left((\partial_x u_\epsilon)^2 + \epsilon \right)^{\frac{p-1}{2}} (\partial_x u_\epsilon)^2 dx, \quad (3.31)$$

we conclude

$$\begin{aligned} &\int_{-\infty}^{\infty} \left((\partial_x u_\epsilon)^2 + \epsilon \right)^{\frac{p-1}{2}} |\partial_x u_\epsilon|^3 dx \\ &\leq C_p \left(\int_{-\infty}^{\infty} |\partial_x u_\epsilon|^{2(p-1)} (\partial_x^2 u_\epsilon)^2 dx \right)^{\frac{1}{3p+1}} \\ &\quad \times \left(\int_{-\infty}^{\infty} \left((\partial_x u_\epsilon)^2 + \epsilon \right)^{\frac{p-1}{2}} (\partial_x u_\epsilon)^2 dx \right)^{\frac{3p+2}{3p+1}} \\ &\leq \frac{\mu}{2} \int_{-\infty}^{\infty} |\partial_x u_\epsilon|^{2(p-1)} (\partial_x^2 u_\epsilon)^2 dx \\ &\quad + C_{p,\mu} \left(\int_{-\infty}^{\infty} \left((\partial_x u_\epsilon)^2 + \epsilon \right)^{\frac{p-1}{2}} (\partial_x u_\epsilon)^2 dx \right)^{\frac{3p+2}{3p}}. \end{aligned} \quad (3.32)$$

Substituting (3.32) and Lemma 3.2 into (3.28), we get

$$\begin{aligned}
& \int_{-\infty}^{\infty} \int_0^{\partial_x u_\epsilon(t)} (s^2 + \epsilon)^{\frac{p-1}{2}} s \, ds dx \\
& + \frac{\mu}{2} \int_0^t \int_{-\infty}^{\infty} \left(\frac{p (\partial_x u_\epsilon)^2 + \epsilon}{(\partial_x u_\epsilon)^2 + \epsilon} \right)^2 \left((\partial_x u_\epsilon)^2 + \epsilon \right)^{p-1} (\partial_x^2 u_\epsilon)^2 \, dx d\tau \\
& \leq \int_{-\infty}^{\infty} \int_0^{\partial_x u_0} (s^2 + \epsilon)^{\frac{p-1}{2}} s \, ds dx \\
& + C_{p,\mu} C_I \left(\sup_{-\tilde{C}-1 \leq u \leq \tilde{C}+1} |(f^\epsilon)''(u)| \right)^{\frac{3p+2}{3p}} \\
& \quad \times \sup_{0 \leq t \leq T} \left(\int_{-\infty}^{\infty} \left((\partial_x u_\epsilon)^2 + \epsilon \right)^{\frac{p-1}{2}} (\partial_x u_\epsilon)^2 \, dx \right)^{\frac{2}{3p}}.
\end{aligned} \tag{3.33}$$

Noting $\frac{2}{3p} < 1$ and

$$\begin{aligned}
& \int_{-\infty}^{\infty} \int_0^{\partial_x u_\epsilon(t)} (s^2 + \epsilon)^{\frac{p-1}{2}} s \, ds dx \\
& \sim \|\partial_x u_\epsilon(t)\|_{L^{p+1}}^{p+1} + \epsilon^{\frac{p-1}{2}} \|\partial_x u_\epsilon(t)\|_{L^2}^2 \\
& \sim \int_{-\infty}^{\infty} \left((\partial_x u_\epsilon)^2 + \epsilon \right)^{\frac{p-1}{2}} |\partial_x u_\epsilon|^3 \, dx,
\end{aligned} \tag{3.34}$$

we can complete the proof of Lemma 3.2.

4. Uniform estimates I. In this section, we show the basic uniform energy estimates with $p > 1$ which is not depending on T , that is, Proposition 3.1. In what follows, we show Proposition 3.1 (also Proposition 3.2 in the next section) provided the solution is sufficiently smooth for simplicity so that we can clearly present the essential process to get the uniform estimates. In order to justify the estimates for the solution obtained in Theorem 3.2, we may take ϵ -regularization again as in Section 3, and take the limit $\epsilon \searrow 0$. Since the process is standard, we omit the details here. We first note the uniform boundedness of ϕ which is proved in Theorem 3.3, that is,

$$\sup_{t \in [0, \infty), x \in \mathbb{R}} |\phi(t, x)| \leq \tilde{C}. \tag{4.1}$$

Now let us rewrite the basic L^2 -energy inequality, that is Proposition 3.1 (uniform estimates I):

$$\begin{aligned}
& \|\phi(t)\|_{L^2}^2 + \int_0^t G(\tau) \, d\tau \\
& + \int_0^t \int_{-\infty}^{\infty} (\partial_x \phi)^2 \left(|\partial_x \phi|^{p-1} + |\partial_x U|^{p-1} + |\partial_x U^r|^{p-1} \right) \, dx d\tau \leq C_p(\phi_0)
\end{aligned} \tag{4.2}$$

for $t \geq 0$, where $G(t)$ is defined as in Proposition 3.1. The proof of (4.2) is given by the following two lemmas.

Lemma 4.1. *It holds that for $t \geq 0$,*

$$\begin{aligned}
& \|\phi(t)\|_{L^2}^2 + \int_0^t G(\tau) \, d\tau \\
& + \int_0^t \int_{-\infty}^{\infty} (\partial_x \phi)^2 \left(|\partial_x \phi|^{p-1} + |\partial_x U|^{p-1} + |\partial_x U^r|^{p-1} \right) \, dx \, d\tau \\
& \leq C_p \|\phi_0\|_{L^2}^2 + C_p \int_0^t (\|\phi(\tau)\|_{L^2}^2 + 1) \left| \int_{-\infty}^{\infty} \left| \widetilde{F}_p(U, U^r) \right| \, dx \right|^{\frac{3p+1}{3p}} (\tau) \, d\tau \\
& + C_p \int_0^t \left| \int_{-\infty}^{\infty} (\partial_x U + \partial_x U^r)^{p-1} (\partial_x U^r)^2 \, dx \right| (\tau) \, d\tau.
\end{aligned}$$

Lemma 4.2. *It holds that*

$$\begin{aligned}
& \int_0^\infty \left| \int_{-\infty}^{\infty} \left| \widetilde{F}_p(U, U^r) \right| \, dx \right|^{\frac{3p+1}{3p}} (t) \, dt < \infty, \\
& \int_0^\infty \left| \int_{-\infty}^{\infty} (\partial_x U + \partial_x U^r)^{p-1} (\partial_x U^r)^2 \, dx \right| (t) \, dt < \infty.
\end{aligned}$$

Once Lemma 4.1 and Lemma 4.2 are proved, by Gronwall's inequality, we have the uniform boundedness

$$\begin{aligned}
& \|\phi(t)\|_{L^2}^2 \leq C_p (\|\phi_0\|_{L^2}^2 + 1) \\
& \times \exp \left\{ \int_0^\infty \left| \int_{-\infty}^{\infty} \left| \widetilde{F}_p(U, U^r) \right| \, dx \right|^{\frac{3p+1}{3p}} \, dt \right\} < \infty
\end{aligned}$$

which easily implies (4.2), that is, Proposition 3.1.

Proof of Lemma 4.1. For $p > 1$, multiplying the equation in (3.7) by ϕ , we obtain the divergence form

$$\begin{aligned}
& \partial_t \left(\frac{1}{2} |\phi|^2 \right) \\
& + \partial_x \left(\phi (f(\tilde{U} + \phi) - f(\tilde{U})) \right) \\
& + \partial_x \left(- \int_{\tilde{U}}^{\tilde{U} + \phi} f(s) \, ds + f(\tilde{U}) \phi \right) \\
& + \partial_x \left(-\mu \phi \left(|\partial_x \tilde{U} + \partial_x \phi|^{p-1} (\partial_x \tilde{U} + \partial_x \phi) - |\partial_x \tilde{U}|^{p-1} \partial_x \tilde{U} \right) \right) \\
& + \left(f(\tilde{U} + \phi) - f(\tilde{U}) - f'(\tilde{U}) \phi \right) \partial_x \tilde{U} \\
& + \mu (\partial_x \phi) \left(|\partial_x \tilde{U} + \partial_x \phi|^{p-1} (\partial_x \tilde{U} + \partial_x \phi) - |\partial_x \tilde{U}|^{p-1} \partial_x \tilde{U} \right) = \phi F_p(U, U^r).
\end{aligned} \tag{4.3}$$

Integrating (4.3) with respect to x and t , we have

$$\begin{aligned} & \frac{1}{2} \|\phi(t)\|_{L^2}^2 + \int_0^t \int_{-\infty}^{\infty} \left(f(\tilde{U} + \phi) - f(\tilde{U}) - f'(\tilde{U})\phi \right) \partial_x \tilde{U} \, dx d\tau \\ & + \mu \int_0^t \int_{-\infty}^{\infty} (\partial_x \phi) \left(|\partial_x \tilde{U} + \partial_x \phi|^{p-1} (\partial_x \tilde{U} + \partial_x \phi) - |\partial_x \tilde{U}|^{p-1} \partial_x \tilde{U} \right) dx d\tau \quad (4.4) \\ & = \frac{1}{2} \|\phi_0\|_{L^2}^2 + \int_0^t \int_{-\infty}^{\infty} \phi F_p(U, U^r) \, dx d\tau. \end{aligned}$$

To estimate the second term on the left-hand side of (4.4), noting the shape of the flux function f , we divide the integral region of x depending on the signs of $\tilde{U} + \phi$, \tilde{U} and ϕ as

$$\begin{aligned} & \int_{-\infty}^{\infty} \left(f(\tilde{U} + \phi) - f(\tilde{U}) - f'(\tilde{U})\phi \right) \partial_x \tilde{U} \, dx \\ & = \int_{-\infty}^{\infty} \left(\int_0^{\phi} \left(\lambda(\tilde{U} + \eta) - \lambda(\tilde{U}) \right) d\eta \right) (\partial_x \tilde{U}) \, dx \quad (4.5) \\ & = \int_{\tilde{U}+\phi \geq 0, \tilde{U} \geq 0, \phi \geq 0} + \int_{\tilde{U}+\phi \geq 0, \tilde{U} \geq 0, \phi \leq 0} + \int_{\tilde{U}+\phi \geq 0, \tilde{U} < 0} + \int_{\tilde{U}+\phi < 0, \tilde{U} \geq 0} \end{aligned}$$

where we used the fact that the integral is clearly zero on the region $\tilde{U} + \phi \leq 0$ and $\tilde{U} \leq 0$. By Lagrange's mean-value theorem, we easily get as

$$\left(\int_{-\infty}^{\infty} \left(\int_0^{\phi} \left(\lambda(\tilde{U} + \eta) - \lambda(\tilde{U}) \right) d\eta \right) (\partial_x \tilde{U}) \, dx \right) (t) \sim G(t) \quad (4.6)$$

where $G = G(t)$ is defined in Proposition 3.1 (cf. [20], [28]). Next, we also estimate the third term on the left-hand side of (4.4) as

$$\begin{aligned} & \int_{-\infty}^{\infty} (\partial_x \phi) \left(|\partial_x \tilde{U} + \partial_x \phi|^{p-1} (\partial_x \tilde{U} + \partial_x \phi) - |\partial_x \tilde{U}|^{p-1} \partial_x \tilde{U} \right) dx \\ & \geq \nu_p^{-1} \int_{-\infty}^{\infty} (\partial_x \phi)^2 \left(|\partial_x \phi|^{p-1} + |\partial_x U|^{p-1} + |\partial_x U^r|^{p-1} \right) dx \quad (4.7) \end{aligned}$$

for some constant $\nu_p > 0$ which is depend only on p . Here, we used the following absolute inequality with $p > 1$, for any $a, b \in \mathbb{R}$,

$$\begin{aligned} & (|a|^{p-1}a - |b|^{p-1}b)(a - b) \\ & \geq C_p^{-1} (|a|^{p-1} + |b|^{p-1})(a - b)^2 \\ & \geq \widetilde{C}_p^{-1} (|a|^{p-1} + |b|^{p-1} + |a - b|^{p-1})(a - b)^2 \end{aligned} \quad (4.8)$$

for some $C_p, \widetilde{C}_p > 0$ depending only on p . Furthermore, we should note

$$\begin{aligned} \left| \int_{-\infty}^{\infty} \phi F_p(U, U^r) \, dx \right| & \leq \int_{-\infty}^{\infty} |\phi| \left| \widetilde{F}_p(U, U^r) \right| dx \\ & + \mu \int_{-\infty}^{\infty} |\partial_x \phi| \left((\partial_x U + \partial_x U^r)^p - (\partial_x U)^p \right) dx. \end{aligned} \quad (4.9)$$

Substituting (4.6), (4.7) and (4.9) into (4.4), we get the energy inequality

$$\begin{aligned}
& \frac{1}{2} \|\phi(t)\|_{L^2}^2 + C_p^{-1} \int_0^t G(\tau) d\tau \\
& + \mu \nu_p^{-1} \int_0^t \int_{-\infty}^{\infty} (\partial_x \phi)^2 \left(|\partial_x \phi|^{p-1} + |\partial_x U|^{p-1} + |\partial_x U^r|^{p-1} \right) dx d\tau \\
& \leq \frac{1}{2} \|\phi_0\|_{L^2}^2 + \int_0^t \int_{-\infty}^{\infty} |\phi| \left| \widetilde{F}_p(U, U^r) \right| dx d\tau \\
& + \mu \int_0^t \int_{-\infty}^{\infty} |\partial_x \phi| \left((\partial_x U + \partial_x U^r)^p - (\partial_x U)^p \right) dx d\tau.
\end{aligned} \tag{4.10}$$

We estimate the second term on the right-hand side of (4.10) as follows:

$$\begin{aligned}
& \int_{-\infty}^{\infty} |\phi| \left| \widetilde{F}_p(U, U^r) \right| dx \\
& \leq C_p \|\phi\|_{L^2}^{\frac{2p}{3p+1}} \|\partial_x \phi\|_{L^{p+1}}^{\frac{p+1}{3p+1}} \int_{-\infty}^{\infty} \left| \widetilde{F}_p(U, U^r) \right| dx \\
& \leq \frac{\mu}{4\nu_p} \|\partial_x \phi\|_{L^{p+1}}^{p+1} + C_p \|\phi\|_{L^2}^{\frac{2}{3}} \left(\int_{-\infty}^{\infty} \left| \widetilde{F}_p(U, U^r) \right| dx \right)^{\frac{3p+1}{3p}} \\
& \leq \frac{\mu}{4\nu_p} \|\partial_x \phi\|_{L^{p+1}}^{p+1} + C_p (\|\phi\|_{L^2}^2 + 1) \left| \int_{-\infty}^{\infty} \left| \widetilde{F}_p(U, U^r) \right| dx \right|^{\frac{3p+1}{3p}},
\end{aligned} \tag{4.11}$$

where we used Young's inequality and the following Sobolev type inequality (cf. [28]):

$$\|\phi\|_{L^\infty} \leq \left(\frac{3p+1}{p+1} \right)^{\frac{p+1}{3p+1}} \|\phi\|_{L^2}^{\frac{2p}{3p+1}} \|\partial_x \phi\|_{L^{p+1}}^{\frac{p+1}{3p+1}}. \tag{4.12}$$

By the Cauchy-Schwarz inequality and Young's inequality, we also estimate the third term on the right-hand side of (4.10) as follows:

$$\begin{aligned}
& \mu \int_{-\infty}^{\infty} |\partial_x \phi| \left((\partial_x U + \partial_x U^r)^p - (\partial_x U)^p \right) dx \\
& = \mu p \int_{-\infty}^{\infty} |\partial_x \phi| \left((\partial_x U + \theta \partial_x U^r)^{p-1} \partial_x U^r \right) dx \\
& \quad (\exists \theta = \theta(t, x) \in (0, 1)) \\
& \leq \frac{\mu}{4\nu_p} \int_{-\infty}^{\infty} (\partial_x \phi)^2 (\partial_x U + \partial_x U^r)^{p-1} dx \\
& \quad + C_p \left| \int_{-\infty}^{\infty} (\partial_x U^r)^2 (\partial_x U + \partial_x U^r)^{p-1} dx \right|.
\end{aligned} \tag{4.13}$$

Thus, substituting (4.11) and (4.13) into (4.10), we complete the proof of Lemma 4.1.

Proof of Lemma 4.2. Firstly, we have

$$\begin{aligned}
& \left| \int_{-\infty}^{\infty} (\partial_x U^r)^2 (\partial_x U + \partial_x U^r)^{p-1} dx \right| \\
& \leq C_p \|\partial_x \tilde{U}(t)\|_{L^\infty}^{p-1} \|\partial_x U^r(t)\|_{L^\infty} \\
& \leq C_p (1+t)^{-1-\frac{p-1}{p+1}},
\end{aligned} \tag{4.14}$$

that is,

$$\int_{-\infty}^{\infty} (\partial_x U^r)^2 (\partial_x U + \partial_x U^r)^{p-1} dx \in L_t^1(0, \infty) \quad (4.15)$$

where we used Lemma 2.2 and Lemma 2.3. Then, it suffices to show, by the definition of the remainder term $\tilde{F}_p(U, U^r)$, that

$$\int_{-\infty}^{\infty} |f'(U + U^r) - f'(U^r)| \partial_x U^r dx \in L_t^{\frac{3p+1}{3p}}(0, \infty), \quad (4.16)$$

$$\int_{-\infty}^{\infty} |f'(U + U^r)| \partial_x U dx \in L_t^{\frac{3p+1}{3p}}(0, \infty). \quad (4.17)$$

To obtain (4.16) and (4.17), it is very natural to divide the integral region \mathbb{R} depending on the sign of $\tilde{U} = U + U^r$. So, for any $t \geq 0$, we introduce

$$X : [0, \infty) \ni t \mapsto X(t) \in \mathbb{R}$$

such that

$$\tilde{U}(t, X(t)) = U(t, X(t)) + U^r(t, X(t)) = 0 \quad (t \geq 0), \quad (4.18)$$

that is,

$$\begin{aligned} U^r(t, X(t)) &= -U(t, X(t)) \\ &= \int_{X(t)}^{\infty} \frac{1}{(1+t)^{\frac{1}{p+1}}} \left(\left(A - B \left(\frac{y}{(1+t)^{\frac{1}{p+1}}} \right)^2 \right) \vee 0 \right)^{\frac{1}{p-1}} dy. \end{aligned} \quad (4.19)$$

Here we note that $X(t)$ uniquely exists because U^r is strictly monotonically increasing with respect to x on the whole \mathbb{R} and U is also strictly monotonically increasing on $-\sqrt{\frac{A}{B}}(1+t)^{\frac{1}{p+1}} < x < \sqrt{\frac{A}{B}}(1+t)^{\frac{1}{p+1}}$. Furthermore, note that $\tilde{U}(t, -\infty) = u_- < 0 < u_+ = \tilde{U}(t, \infty)$. Therefore we can divide the integral region \mathbb{R} into $(-\infty, X(t))$ where $\tilde{U} < 0$ and $(X(t), \infty)$ where $\tilde{U} > 0$. As a basic behavior of $X(t)$, we can show by Lemma 2.2 and Lemma 2.3 that there exists a positive time T_0 such that for some $\delta \in (0, \sqrt{\frac{A}{B}})$,

$$\left(\sqrt{\frac{A}{B}} - \delta \right) (1+t)^{\frac{1}{p+1}} < X(t) < \sqrt{\frac{A}{B}} (1+t)^{\frac{1}{p+1}} \quad (t \geq T_0). \quad (4.20)$$

Indeed, by an easy fact

$$\sup_{x \in \mathbb{R}} \left| u^r \left(\frac{x}{1+t} \right) - u^r \left(\frac{x}{t} \right) \right| \leq C(1+t)^{-1}, \quad (4.21)$$

and Lemma 2.2, it follows that

$$\sup_{x \in \mathbb{R}} \left| U^r(t, x) - u^r \left(\frac{x}{1+t} \right) \right| \leq C_{\epsilon}(1+t)^{-1+\epsilon} \quad (\epsilon \in (0, 1)), \quad (4.22)$$

which implies

$$\begin{aligned} \lim_{t \rightarrow \infty} \tilde{U} \left(t, \left(\sqrt{\frac{A}{B}} - \delta \right) (1+t)^{\frac{1}{p+1}} \right) \\ = - \int_{\sqrt{\frac{A}{B}} - \delta}^{\infty} \left((A - B \xi^2) \vee 0 \right)^{\frac{1}{p-1}} d\xi < 0, \end{aligned} \quad (4.23)$$

and

$$\tilde{U} \left(t, \sqrt{\frac{A}{B}} (1+t)^{\frac{1}{p+1}} \right) = U^r \left(t, \sqrt{\frac{A}{B}} (1+t)^{\frac{1}{p+1}} \right) > 0 \quad (\forall t \geq 0). \quad (4.24)$$

So we have (4.20) by (4.23) and (4.24). Then, by (4.18), (4.21) and (4.22), we have for any $\epsilon \in (0, 1)$, there exists a positive constant C_ϵ such that

$$\left| (\lambda)^{-1} \left(\frac{X(t)}{1+t} \right) - \int_{\frac{X(t)}{(1+t)^{\frac{1}{p+1}}}}^{\infty} \left((A - B \xi^2) \vee 0 \right)^{\frac{1}{p-1}} d\xi \right| \leq C_\epsilon (1+t)^{-1+\epsilon}, \quad (4.25)$$

for $t \geq T_0$. Using (4.25), we can show more precise large time behavior of $X(t)$ as in the following lemma.

Lemma 4.3. *It holds that for each $p > 1$, there exists a positive constant C_p such that*

$$\left| \sqrt{\frac{A}{B}} - \frac{X(t)}{(1+t)^{\frac{1}{p+1}}} \right| \leq C_p (1+t)^{-\frac{p-1}{p+1}} \quad (t \geq T_0).$$

Let us admit Lemma 4.3 for a moment and complete the proof of Lemma 4.2. We shall give the proof of Lemma 4.3 after the proof of Lemma 4.2. Using Lemmas 2.2, 2.3 and 4.3, we first prove (4.16). Dividing the integral region as we mentioned above as

$$\int_{-\infty}^{\infty} |f'(U + U^r) - f'(U^r)| \partial_x U^r dx = \int_{-\infty}^{X(t)} + \int_{X(t)}^{\infty} =: I_{11} + I_{12},$$

we estimate each integral as follows:

$$\begin{aligned} I_{11}(t) &= \int_{-\infty}^{X(t)} |f'(U + U^r) - f'(U^r)| \partial_x U^r dx \\ &= \int_{-\infty}^{X(t)} \partial_x (f(U^r)) dx \\ &\leq C |U^r(t, X(t))|^2 \\ &\leq C \left(\frac{X(t)}{1+t} + C_\epsilon (1+t)^{-1+\epsilon} \right)^2 \\ &\leq C_p (1+t)^{-1-\frac{p-1}{p+1}} + C_\epsilon (1+t)^{-1-(1-2\epsilon)} \quad (\epsilon \in (0, 1), t \geq 0), \end{aligned} \quad (4.26)$$

$$\begin{aligned}
I_{12}(t) &= \int_{X(t)}^{\infty} |f'(U + U^r) - f'(U^r)| \partial_x U^r dx \\
&\leq C \int_{X(t)}^{\infty} |U| \partial_x U^r dx \\
&\leq C(1+t)^{-1} \int_{X(t)}^{\infty} \int_{\frac{x}{(1+t)^{\frac{1}{p+1}}}}^{\infty} \left((A - B\xi^2) \vee 0 \right)^{\frac{1}{p-1}} d\xi dx \\
&= C(1+t)^{-1} \int_{\frac{X(t)}{(1+t)^{\frac{1}{p+1}}}}^{\infty} \left((A - B\xi^2) \vee 0 \right)^{\frac{1}{p-1}} \left(\xi(1+t)^{\frac{1}{p+1}} - X(t) \right) d\xi \\
&\leq C_p(1+t)^{-1+\frac{1}{p+1}} \int_{\frac{X(t)}{(1+t)^{\frac{1}{p+1}}}}^{\sqrt{\frac{A}{B}}} \xi (A - B\xi^2)^{\frac{1}{p-1}} d\xi \\
&= C_p(1+t)^{-1+\frac{1}{p+1}} \cdot \frac{p-1}{2Bp} \left(A - B \left(\frac{X(t)}{(1+t)^{\frac{1}{p+1}}} \right)^2 \right)^{\frac{p}{p-1}} \\
&\leq C_p(1+t)^{-1+\frac{1}{p+1}} \left| \sqrt{\frac{A}{B}} - \frac{X(t)}{(1+t)^{\frac{1}{p+1}}} \right|^{\frac{p}{p-1}} \\
&\leq C_p(1+t)^{-1-\frac{p-1}{p+1}} \quad (t \geq T_0),
\end{aligned} \tag{4.27}$$

where we used the facts $\|\partial_x U^r(t)\|_{L^\infty} \leq C(1+t)^{-1}$ in Lemma 2.2 and Lemma 4.3. Hence, choosing ϵ suitably small in (4.26), we can easily conclude $I_{11}, I_{12} \in L_t^{\frac{3p+1}{3p}}(0, \infty)$, which proves (4.16). Next, we similarly show (4.17). In this case, noting

$$\int_{-\infty}^{\infty} f'(U + U^r) \partial_x U dx = \int_{X(t)}^{\infty} f'(U + U^r) \partial_x U dx =: I_{21},$$

we estimate I_{21} , by integration by parts, as follows:

$$\begin{aligned}
I_{21}(t) &= \int_{X(t)}^{\infty} f'(U + U^r) \partial_x U dx \\
&= - \int_{X(t)}^{\infty} U f''(U + U^r) (\partial_x U + \partial_x U^r) dx \\
&\leq C \int_{X(t)}^{\infty} -\frac{1}{2} \partial_x (U^2) dx + C \int_{X(t)}^{\infty} |U| \partial_x U^r dx \\
&\leq C |U^r(t, X(t))|^2 + C_p(1+t)^{-1-\frac{p-1}{p+1}} \\
&\leq C_\epsilon(1+t)^{-1-(1-2\epsilon)} + C_p(1+t)^{-1-\frac{p-1}{p+1}} \quad (\epsilon \in (0, 1), t \geq T_0).
\end{aligned}$$

Hence, choosing ϵ suitably small again, we easily have $I_{21} \in L^{\frac{3p+1}{3p}}(0, \infty)$. Thus, the proof of Lemma 4.2 is complete.

Proof of Lemma 4.3. First, we note from (4.20) that

$$\left(\sqrt{\frac{A}{B}} - \delta \right) (1+t)^{-\frac{p}{p+1}} < \frac{X(t)}{1+t} < \sqrt{\frac{A}{B}} (1+t)^{-\frac{p}{p+1}} \quad (t \geq T_0). \quad (4.28)$$

Then, by using (4.25), we have for any $\epsilon \in (0, 1)$, there exists a positive constant C_ϵ such that

$$\begin{aligned} & (\lambda)^{-1} \left(\sqrt{\frac{A}{B}} (1+t)^{-\frac{p}{p+1}} \right) + C_\epsilon (1+t)^{-1+\epsilon} \\ & \geq (\lambda)^{-1} \left(\frac{X(t)}{1+t} \right) + C_\epsilon (1+t)^{-1+\epsilon} \\ & \geq B^{\frac{1}{p+1}} \int_{\frac{X(t)}{(1+t)^{\frac{1}{p+1}}}}^{\sqrt{\frac{A}{B}}} \left(\frac{A}{B} - \xi^2 \right)^{\frac{1}{p-1}} d\xi \\ & \geq (AB)^{\frac{1}{2(p-1)}} \int_{\frac{X(t)}{(1+t)^{\frac{1}{p+1}}}}^{\sqrt{\frac{A}{B}}} \left(\sqrt{\frac{A}{B}} - \xi \right)^{\frac{1}{p-1}} d\xi \quad (t \geq T_0). \end{aligned} \quad (4.29)$$

Hence, by taking $\epsilon = \frac{1}{p+1}$, it implies that for $t \geq T_0$,

$$C_p (1+t)^{-\frac{p}{p+1}} \geq (AB)^{\frac{1}{2(p-1)}} \left(\frac{p-1}{p} \right) \left| \sqrt{\frac{A}{B}} - \frac{X(t)}{(1+t)^{\frac{1}{p+1}}} \right|^{\frac{p}{p-1}} \quad (4.30)$$

which completes the proof of Lemma 4.4.

Thus, we do complete the proof of Proposition 3.1.

Remark 4.1. The unique global solution in time u also satisfies the following regularity

$$\begin{cases} \partial_t u \in L^\infty(0, T; L^{p+1}) \cap \left(L^{p+1}(0, T; L^{p+1}) \oplus L^2(0, T; L^2) \right), \\ \lambda(u) \partial_x u \in L^\infty(0, T; L^{p+1}). \end{cases}$$

5. Uniform estimates II. In this section, In order to complete the uniform estimates for the asymptotics not depending on T , we show Proposition 3.2. To do that, we assume that the solution to our Cauchy problem (3.8) satisfies the same regularity as in Section 4. What we should prove is the following energy inequality:

$$\begin{aligned} & \|\partial_x u(t)\|_{L^{p+1}}^{p+1} + \int_0^t \int_{-\infty}^\infty |\partial_x u|^{2(p-1)} (\partial_x^2 u)^2 dx d\tau \\ & + \int_0^t \|\partial_x u(\tau)\|_{L^{p+2}(\{x \in \mathbb{R} \mid u > 0\})}^{p+2} d\tau \leq C_p(\phi_0, \partial_x u_0) \quad (t \geq 0). \end{aligned} \quad (5.1)$$

In order to obtain (5.1), we multiple the equation in the original Cauchy problem (1.1) (not the reformulated problem, that is (3.7) or (3.8)) by

$$-\partial_x \left(|\partial_x u|^{q-1} \partial_x u \right)$$

with $q > 1$ and obtain the divergence form

$$\begin{aligned} & \partial_t \left(\frac{1}{q+1} |\partial_x u|^{q+1} \right) + \partial_x \left(- |\partial_x u|^{q-1} \partial_x u \cdot \partial_t u \right) \\ & + \partial_x \left(- \frac{q}{q+1} f'(u) |\partial_x u|^{q+1} \right) \\ & + \frac{q}{q+1} f''(u) |\partial_x u|^{q+1} \partial_x u + \mu p q |\partial_x u|^{p+q-2} (\partial_x^2 u)^2 = 0. \end{aligned} \quad (5.2)$$

Integrating the divergence form (5.2) with respect to x , we have

$$\begin{aligned} & \frac{1}{q+1} \frac{d}{dt} \|\partial_x u(t)\|_{L^{q+1}}^{q+1} + \mu p q \int_{-\infty}^{\infty} |\partial_x u|^{p+q-2} (\partial_x^2 u)^2 dx \\ & + \frac{q}{q+1} \int_{-\infty}^{\infty} f''(u) |\partial_x u|^{q+1} \partial_x u dx = 0. \end{aligned} \quad (5.3)$$

Now we separate the integral region to the third term on the left-hand side of (5.3) as

$$\begin{aligned} & \int_{-\infty}^{\infty} f''(u) |\partial_x u|^{q+1} \partial_x u dx \\ & = \int_{\partial_x u \geq 0} + \int_{\partial_x u < 0} \\ & = \int_{\partial_x u \geq 0} f''(u) |\partial_x u|^{q+2} dx - \int_{\partial_x u < 0} f''(u) |\partial_x u|^{q+2} dx. \end{aligned} \quad (5.4)$$

Substituting (5.4) into (5.3), we get the following equality

$$\begin{aligned} & \frac{1}{q+1} \frac{d}{dt} \|\partial_x u(t)\|_{L^{q+1}}^{q+1} + \mu p q \int_{-\infty}^{\infty} |\partial_x u|^{p+q-2} (\partial_x^2 u)^2 dx \\ & + \frac{q}{q+1} \int_{\partial_x u \geq 0} f''(u) |\partial_x u|^{q+2} dx = \frac{q}{q+1} \int_{\partial_x u < 0} f''(u) |\partial_x u|^{q+2} dx. \end{aligned} \quad (5.5)$$

We have the following result which plays the most important role in the proof of (5.1).

Lemma 5.1. *For each $q > 1$, there exists a positive constant C_q such that*

$$\int_{\partial_x u < 0} f''(u) |\partial_x u|^{q+2} dx \leq C_q \int_{\partial_x u < 0} |\partial_x \phi|^{q+2} dx. \quad (5.6)$$

In fact, taking care of the relation

$$\partial_x u = \partial_x \tilde{U} + \partial_x \phi < 0 \iff \partial_x \phi < 0, \partial_x \tilde{U} < |\partial_x \phi|, \quad (5.7)$$

we immediately have

$$\begin{aligned} & \int_{\partial_x u < 0} f''(u) |\partial_x u|^{q+2} dx \\ & \leq 2^{q+2} \left(\sup_{0 \leq u \leq \tilde{C}+1} f''(u) \right) \int_{\partial_x \phi < 0, \partial_x \tilde{U} < |\partial_x \phi|} |\partial_x \phi|^{q+2} dx. \end{aligned} \quad (5.8)$$

Remark 5.1. Under the relation (5.7), noting

$$\int_0^\infty \int_{-\infty}^\infty |\partial_x \phi|^{p+1} dx dt < \infty$$

from (4.1) in Section 4 and taking $q = p - 1$ to (5.5), we can easily show that for $p \geq \frac{3}{2}$,

$$\begin{aligned} & \frac{1}{p} \|\partial_x u(t)\|_{L^p}^p + \mu p(p-1) \int_0^\infty \int_{-\infty}^\infty |\partial_x u|^{2p-3} (\partial_x^2 u)^2 dx dt \\ & + \frac{p-1}{p} \int_0^\infty \int_{\partial_x u \geq 0} f''(u) |\partial_x u|^{p+1} dx dt < \infty, \end{aligned} \quad (5.9)$$

which namely means that for $p \geq \frac{3}{2}$,

$$\begin{cases} \frac{d}{dt} \|\partial_x u\|_{L^p}^p \in L_t^1(0, \infty), \\ \int_{-\infty}^\infty |\partial_x u|^{2p-3} (\partial_x^2 u)^2 dx \in L_t^1(0, \infty), \\ \int_{-\infty}^\infty f''(u) |\partial_x u|^{p+1} dx \sim \int_{u>0} |\partial_x u|^{p+1} dx \in L_t^1(0, \infty). \end{cases} \quad (5.10)$$

Integrating (5.5) with respect to t and taking $q = p$, we have the energy equality

$$\begin{aligned} & \frac{1}{p+1} \|\partial_x u(t)\|_{L^{p+1}}^{p+1} + \mu p^2 \int_0^t \int_{-\infty}^\infty |\partial_x u|^{2(p-1)} (\partial_x^2 u)^2 dx d\tau \\ & + \frac{p}{p+1} \int_0^t \int_{\partial_x u \geq 0} f''(u) |\partial_x u|^{p+2} dx d\tau \\ & = \frac{1}{p+1} \|\partial_x u_0\|_{L^{p+1}}^{p+1} + \frac{p}{p+1} \int_0^t \int_{\partial_x u < 0} f''(u) |\partial_x u|^{p+2} dx d\tau. \end{aligned} \quad (5.11)$$

The most difficult term to estimate is the second term on the right-hand side. We prepare the following “boundary zero condition type” interpolation inequality to overcome the difficulty.

Lemma 5.2. *It holds that*

$$\begin{aligned} & \int_{\partial_x u < 0} |\partial_x u|^{p+2} dx \\ & \leq C_p \left(\int_{\partial_x u < 0} |\partial_x u|^{2(p-1)} (\partial_x^2 u)^2 dx \right)^{\frac{1}{3p+1}} \left(\int_{\partial_x u < 0} |\partial_x u|^{p+1} dx \right)^{\frac{3p+2}{3p+1}}. \end{aligned} \quad (5.12)$$

Proof of Lemma 5.2. Since $\partial_x u$ is absolutely continuous, we first note that for any $x \in \{x \in \mathbb{R} \mid \partial_x u < 0\}$, there exists $x_k \in \mathbb{R} \cup \{-\infty\}$ such that

$$\partial_x u(x_k) = 0, \quad \partial_x u(y) < 0 \quad (y \in (x_k, x)).$$

Therefore by using the Cauchy-Schwarz inequality, it follows that for such x and x_k with $q \geq p (> 1)$,

$$\begin{aligned} |\partial_x u|^q &= (-\partial_x u)^q = q \int_{x_k}^x (-\partial_x u)^{q-1} (-\partial_x^2 u) \, dy \\ &\leq q \int_{\partial_x u < 0} (-\partial_x u)^{q-1} (-\partial_x^2 u) \, dx \\ &\leq q \left(\int_{\partial_x u < 0} (-\partial_x u)^{2(p-1)} (-\partial_x^2 u)^2 \, dx \right)^{\frac{1}{2}} \left(\int_{\partial_x u < 0} (-\partial_x u)^{2(q-p)} \, dx \right)^{\frac{1}{2}}. \end{aligned} \quad (5.13)$$

Hence

$$\begin{aligned} \|\partial_x u(t)\|_{L_x^\infty(\{\partial_x u < 0\})} &\leq q^{\frac{1}{q}} \left(\int_{\partial_x u < 0} |\partial_x u|^{2(p-1)} (\partial_x^2 u)^2 \, dx \right)^{\frac{1}{2q}} \left(\int_{\partial_x u < 0} |\partial_x u|^{2(q-p)} \, dx \right)^{\frac{1}{2q}}. \end{aligned} \quad (5.14)$$

So we get

$$\begin{aligned} &\int_{\partial_x u < 0} |\partial_x u|^{p+2} \, dx \\ &\leq \|\partial_x u\|_{L_x^\infty(\{\partial_x u < 0\})} \int_{\partial_x u < 0} |\partial_x u|^{p+1} \, dx \\ &\leq q^{\frac{1}{q}} \left(\int_{\partial_x u < 0} |\partial_x u|^{2(p-1)} (\partial_x^2 u)^2 \, dx \right)^{\frac{1}{2q}} \\ &\quad \times \left(\int_{\partial_x u < 0} |\partial_x u|^{2(q-p)} \, dx \right)^{\frac{1}{2q}} \left(\int_{\partial_x u < 0} |\partial_x u|^{p+1} \, dx \right). \end{aligned} \quad (5.15)$$

Taking $q = \frac{3p+2}{2}$ in (5.15), we have

$$\begin{aligned} \int_{\partial_x u < 0} |\partial_x u|^{p+2} \, dx &\leq \left(\frac{3p+2}{2} \right)^{\frac{2}{3p+2}} \left(\int_{\partial_x u < 0} |\partial_x u|^{p+1} \, dx \right)^{\frac{3p+2}{3p+1}} \\ &\quad \times \left(\int_{\partial_x u < 0} |\partial_x u|^{2(p-1)} (\partial_x^2 u)^2 \, dx \right)^{\frac{1}{3p+1}}. \end{aligned} \quad (5.16)$$

Thus we complete the proof.

Using Young's inequality to Lemma 5.2, (5.12), we also have

Lemma 5.3. *It follows that for any $\epsilon > 0$, there exists a positive constant $C_p(\epsilon)$ such that,*

$$\begin{aligned} &\int_{\partial_x u < 0} |\partial_x u|^{p+2} \, dx \\ &\leq \epsilon \int_{\partial_x u < 0} |\partial_x u|^{2(p-1)} (\partial_x^2 u)^2 \, dx + C_p(\epsilon) \left(\int_{\partial_x u < 0} |\partial_x u|^{p+1} \, dx \right)^{\frac{3p+2}{3p}}. \end{aligned} \quad (5.17)$$

Substituting (5.17) with $\epsilon = \frac{\mu p^2}{2}$ into (5.11), we have

$$\begin{aligned} & \frac{1}{p+1} \|\partial_x u(t)\|_{L^{p+1}}^{p+1} + \frac{\mu p^2}{2} \int_0^t \int_{-\infty}^{\infty} |\partial_x u|^{2(p-1)} (\partial_x^2 u)^2 dx d\tau \\ & + \frac{p}{p+1} \int_0^t \int_{\partial_x u \geq 0} f''(u) |\partial_x u|^{p+2} dx d\tau \\ & \leq \frac{1}{p+1} \|\partial_x u_0\|_{L^{p+1}}^{p+1} + C_p \int_0^t \left(\int_{\partial_x u < 0} |\partial_x u|^{p+1} dx \right)^{\frac{2}{3p}+1} d\tau. \end{aligned} \quad (5.18)$$

Now recalling Lemma 5.1, we have

$$\int_{\partial_x u < 0} |\partial_x u|^{p+1} dx \leq C_p \int_{-\infty}^{\infty} |\partial_x \phi|^{p+1} dx \in L_t^1(0, \infty). \quad (5.19)$$

We also note $\frac{2}{3p} < 1$ and focus on the fact

$$\left(\int_{\partial_x u < 0} |\partial_x u|^{p+1} dx \right)^{\frac{2}{3p}} \leq C_p \left(1 + \int_{\partial_x u < 0} |\partial_x u|^{p+1} dx \right) \quad (5.20)$$

for some positive constant C_p . Hence, substituting (5.19) and (5.20) into (5.18), we have

$$\begin{aligned} & \frac{1}{p+1} \|\partial_x u(t)\|_{L^{p+1}}^{p+1} + \frac{\mu p^2}{2} \int_0^t \int_{-\infty}^{\infty} |\partial_x u|^{2(p-1)} (\partial_x^2 u)^2 dx d\tau \\ & + \frac{p}{p+1} \int_0^t \int_{\partial_x u \geq 0} f''(u) |\partial_x u|^{p+2} dx d\tau \\ & \leq \frac{1}{p+1} \|\partial_x u_0\|_{L^{p+1}}^{p+1} + C_p \int_0^t \int_{-\infty}^{\infty} |\partial_x \phi|^{p+1} dx d\tau \\ & \quad + C_p \int_0^t \|\partial_x u(\tau)\|_{L^{p+1}}^{p+1} \left(\int_{-\infty}^{\infty} |\partial_x \phi|^{p+1} dx \right) d\tau. \end{aligned} \quad (5.21)$$

By using Gronwall's inequality, we have

$$\begin{aligned} \|\partial_x u(t)\|_{L^{p+1}}^{p+1} & \leq C_p (\|\partial_x u_0\|_{L^{p+1}}^{p+1} + \|\phi_0\|_{L^2}^2 + 1) \\ & \quad \times \exp \left\{ C_p \int_0^t \int_{-\infty}^{\infty} |\partial_x \phi|^{p+1} dx dt \right\} < \infty. \end{aligned} \quad (5.22)$$

Hence, substituting (5.22) into (5.21), it finally holds

$$\begin{aligned} & \|\partial_x u(t)\|_{L^{p+1}}^{p+1} + \int_0^t \int_{-\infty}^{\infty} |\partial_x u|^{2(p-1)} (\partial_x^2 u)^2 dx d\tau \\ & + \int_0^t \|\partial_x u(\tau)\|_{L^{p+2}(\{x \in \mathbb{R} \mid u > 0\})}^{p+2} d\tau \\ & \leq C (\|\phi_0\|_{L^2}, \|\partial_x u_0\|_{L^{p+1}}). \end{aligned} \quad (5.23)$$

Thus, we do complete the proof of Proposition 3.2.

6. Asymptotic behavior. In this section, we shall obtain the asymptotic behavior

$$\sup_{x \in \mathbb{R}} |\phi(t, x)| \xrightarrow{t \rightarrow \infty} 0 \quad (6.1)$$

by utilizing Proposition 3.1, (3.11) and Proposition 3.2, (3.12). Noting the Gagliardo-Nirenberg inequality (cf. [3], [23], [24]), we have

$$\begin{aligned} & \sup_{x \in \mathbb{R}} |\phi(t, x)| \\ & \leq C_p \|\phi(t)\|_{L^2}^{\frac{2p}{3p+1}} \|\partial_x \phi(t)\|_{L^{p+1}}^{\frac{p+1}{3p+1}} \\ & \leq C_p (\|\phi_0\|_{L^2}^2 + 1)^{\frac{2p}{3p+1}} \left(\|\partial_x u(t)\|_{L^{p+1}}^{\frac{p+1}{3p+1}} + \|\partial_x \tilde{U}(t)\|_{L^{p+1}}^{\frac{p+1}{3p+1}} \right), \end{aligned} \quad (6.2)$$

and we also note

$$\|\partial_x \tilde{U}(t)\|_{L^{p+1}} \leq C_p (1+t)^{-\frac{1}{p+1}} \xrightarrow{t \rightarrow \infty} 0. \quad (6.3)$$

Hence, it suffices to prove

Lemma 6.1. *It holds that*

$$\|\partial_x u(t)\|_{L^{p+1}} \xrightarrow{t \rightarrow \infty} 0. \quad (6.4)$$

To show Lemma 6.1, we claim

Lemma 6.2. *It holds that*

$$\frac{d}{dt} \|\partial_x u\|_{L^{p+1}}^{p+1} \in L_t^1(0, \infty). \quad (6.5)$$

Once it holds, Lemma 6.1 immediately follows. In fact, for any sequence

$$\{t_k\}_{k=1}^\infty \subset [0, \infty) \quad (t_k \nearrow \infty \text{ as } k \rightarrow \infty),$$

we have

$$\begin{aligned} & \left| \|\partial_x u(t_m)\|_{L^{p+1}}^{p+1} - \|\partial_x u(t_n)\|_{L^{p+1}}^{p+1} \right| \\ & \leq \left| \int_0^{t_m} \frac{d}{dt} \|\partial_x u(t)\|_{L^{p+1}}^{p+1} dt - \int_0^{t_n} \frac{d}{dt} \|\partial_x u(t)\|_{L^{p+1}}^{p+1} dt \right| \\ & \xrightarrow{m, n \rightarrow \infty} 0. \end{aligned} \quad (6.6)$$

Therefore $\{\|\partial_x u(t_k)\|_{L^{p+1}}^{p+1}\}_{k=1}^\infty \subset \mathbb{R}$ is a Cauchy sequence which has a limit α in \mathbb{R} . Because $\{t_k\}_{k=1}^\infty$ is arbitrarily taken, the limit α should be independent of $\{t_k\}_{k=1}^\infty$ satisfying

$$\alpha = \lim_{t \rightarrow \infty} \|\partial_x u(t)\|_{L^{p+1}}^{p+1} = \lim_{t \rightarrow \infty} \|\partial_x \phi(t)\|_{L^{p+1}}^{p+1}. \quad (6.7)$$

Thus, it holds $\alpha = 0$ by $\|\partial_x \phi\|_{L^{p+1}}^{p+1} \in L_t^1(0, \infty)$.

It remains to prove Lemma 6.2.

Proof of Lemma 6.2. Direct calculation shows that

$$\begin{aligned} & \frac{d}{dt} \int_{-\infty}^{\infty} |\partial_x u|^{p+1} dx \\ &= (p+1) \int_{-\infty}^{\infty} \left(|\partial_x u|^{p-1} \partial_x u \right) \partial_x (\partial_t u) dx \\ &= - \int_{-\infty}^{\infty} \left(p f''(u) |\partial_x u|^{p+1} \partial_x u + \mu p^2 (p+1) |\partial_x u|^{2(p-1)} (\partial_x^2 u)^2 \right) dx. \end{aligned} \quad (6.8)$$

Integrating (6.8) with respect to t , we have by (3.11) and (3.12),

$$\begin{aligned} & \int_0^{\infty} \left| \frac{d}{dt} \int_{-\infty}^{\infty} |\partial_x u|^{p+1} dx \right| dt \\ & \leq p \left(\sup_{0 \leq u \leq \bar{C}+1} f''(u) \right) \int_0^{\infty} \int_{u>0} |\partial_x u|^{p+2} dx dt \\ & \quad + \mu p^2 (p+1) \int_0^{\infty} \int_{-\infty}^{\infty} |\partial_x u|^{2(p-1)} (\partial_x^2 u)^2 dx dt \\ & \leq C_p (\|\phi_0\|_{L^2}, \|\partial_x u_0\|_{L^{p+1}}). \end{aligned} \quad (6.9)$$

Hence, we complete the proof of Lemma 6.1.

Thus we have obtained the asymptotic behavior (6.1).

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